Multifractal Spectrum of Branching Random Walks on Free Groups

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Backgrounds

Free Group generated by $\{a, b\}$.

(symmetric) generating set $S := \{a, b, a^{-1}, b^{-1}\}$. $\mathbb{F}^2 := \{x_1 x_2 \cdots x_n : n \ge 0, x_{i+1} \ne x_i^{-1} \forall 1 \le i \le n\}.$



Random walk on Free Group



- Let Y₁, Y₂, · · · , Y_n be sampled independently according to a distribution μ on the alphabet {a, b, a⁻¹, b⁻¹, e}.
- Set $Z_0 = e$ and

$$Z_n := Y_1 \cdots Y_n, n \ge 1$$

We call $(Z_n)_{n\geq}$ a (nearest neighbor) random walk on \mathbb{F}^2 .

• μ is called symmetric if $\mu(x) = \mu(x^{-1})$.

Random walk on Z^2

If our symbol a, b satisfies an additional relation

ab = ba

Then use $\{a, b\}$ to generate a group, we get \mathbb{Z}^2 .



Random Walk on Group

Assume that (Γ, S) is a group with generating set S.

Let Y_1, \dots, Y_n be independently sampled according to certain probability measure μ on S. Then

$$Z_n = Y_1 \cdots, Y_n$$

defines a random walk on the group (Γ, S) .



Figure 1: $\pi_1(\Sigma_2)$, the fundamental group of the a genus-2 torus

In the previous examples, the groups \mathbb{F}^2 and $\pi_1(\Sigma_2)$ are similar, while \mathbb{Z}^2 are very different from them:

- The size of a ball of radius n is of order exp(Θ(n)) in F² and π₁(Σ₂).
- Consider a simple random walk on \mathbb{F}^2 and $\pi_1(\Sigma_2)$, and and denote $p_n(u, v)$ its transition probability, then

$$||P|| = \lim_{n \to \infty} p_{2n}(v, v)^{1/2n} < 1.$$

• The geodesic triangles in \mathbb{F}^2 and $\pi_1(\Sigma_2)$ are thin:



Thin and not thin triangles

The so called (non-elementary) hyperbolic group is a class of group that generalizes \mathbb{F}^2 and $\pi_1(\Sigma_2)$.

A branching random walk (BRW) on a group Γ with a finite symmetric generating set S is constructed as follows:

- Sample a Galton-Watson tree *T* with offspring distribution (*p_k*)_{k≥0} and let *r* := ∑_{k≥1} *kp_k*.
- Independently assign each non-root vertex *u* a random element
 Y_u ∈ *S* according to a symmetric probability measure μ on *S* ∪ {*e*}.
- Define V(u) := Y_{u1}Y_{u2} ··· Y_{un} where (root, u1, ··· , un = u) is the geodesic in *T* from root to u. Moreover set V(root) := identity in Γ.

Technical assumption: $\sum_k e^{sk} p_k < \infty$ for some s > 0.

Double phase transition

BRWs on non-elementary hyperbolic groups (and generally, nonamenable groups) exhibits a double phase transition that does not occur in the corresponding processes on Euclidean spaces.

Benjamini–Peres' 94 and **Gantert–Müller' 06**: For a BRW (r, μ) , let

$$R := \|P\|^{-1} = \frac{1}{\limsup_{n \to \infty} p_{2n}(v, v)^{1/2n}} > 1.$$

- If $0 \le r \le 1$ then the process dies after finite time,
- If 1 < r ≤ R then does not visit any compact set infinitely often. (Transient/weak survival regime)
- r > R the BRW visits every vertex infinitely often when it survives forever. (recurrent/strong survival regime)

The ICM 2006 lecture by **Lalley**: The weak/strong survival transition on trees and nonamenable graphs.

The boundary of a free group

The free group \mathbb{F}^d admits a natural boundary:

 $\partial \mathbb{F} :=$ the collection of all geodesic rays starting from the root.

For $x, y \in \overline{\mathbb{F}} := \mathbb{F} \cup \partial \mathbb{F}$, the standard ultra-distance metric between x, y is defined as

$$\operatorname{dist}_{\bar{\mathbb{F}}}(x,y) := e^{-|x \wedge y|}$$

Then one have $\dim_{\mathrm{H}}(\partial \mathbb{F}^d) = \ln(2d-1)$.



For a <u>BRW on \mathbb{F} </u> with mean *r* and step distribution μ , define the limit set

 $\Lambda_r := \{x \in \mathbb{F} : x \text{ is visited by the BRW}\}.$

In the transient/weak survival regime, Λ_r is a (random) proper subset of $\partial \mathbb{F}$. Otherwise it is either empty or $\partial \mathbb{F}$.

Liggett'96 and Hueter-Lalley'00

They obtained the Hausdorff dimension of the limit set Λ_r . Specifically, for $r \in (1, R]$,

$$\dim_{\mathrm{H}}(\Lambda_r) = \ln G(r),$$

where G(r) is the growth rate of the trace of the BRW

$$G(r) := \lim_{n \to \infty} |\{x \in \mathbb{F} : |x| = n, x \text{ is visited by the BRW }\}|^{\frac{1}{n}}$$
 a.s.

Moreover G(r) is the solution of the following equation $\frac{F_a(r)}{\rho + F_a(r)} + \frac{F_b(r)}{\rho + F_b(r)} = \frac{1}{2}$ with $F_a(r) := \sum_{n=1}^{\infty} \mathbb{P}(T_a = n)r^n$ and T_a is the first passage time of the RW.

Interesting phenomenon happens at critical case:

Hueter–Lalley'00: backscattering inequality For $r \in (1, R]$, $\dim_{\mathrm{H}}(\Lambda_r) \leq \frac{1}{2} \dim_{\mathrm{H}}(\partial \mathbb{F}).$ Above, the equality holds if and only if

r = R and μ is isotropic.

• How come $\frac{1}{2}$ appears?

BRW on hyperbolic groups

- For a hyperbolic group Γ, it has a natural boundary ∂Γ called Gromov boundary, equipped with the visual metric.
- Thus limit set Λ_r is also well-defined for BRW on hyperbolic groups.

Sidoravicius–Wang–Xiang'23

Consider BRW on hyperbolic groups. For $r \in (1, R)$,

 $\dim_{\mathrm{H}}(\Lambda_r) \propto \ln G(r),$

and

$$\dim_{\mathrm{H}}(\Lambda_r) \leq \frac{1}{2} \dim_{\mathrm{H}}(\partial \Gamma).$$

Dussaule–Wang–Yang'25

For BRW on relative hyperbolic groups with $r \in (1, R]$, the above holds.

Monofractal: The irregularity of the object is the same at every point.

An example of monofractal: standard Brownian motion.

Brownian motion is a monofractal because its behavior is (to first order at least) described by a single fractal exponent 1/2.

One way to say this is

- Almost surely, BM is everywhere locally (¹/₂ − ε)- Hölder continuous; and at every point, BM fails to be locally (¹/₂ + ε)-Hölder continuous.
- For every point t,

$$\mathbb{E}[(B(t+\epsilon)-B(t))^q] \asymp \epsilon^{\frac{1}{2}q}$$

An example of multifractal: 2D Gaussian Free Field h.

A point z is called α -thick if

$$\liminf_{\varepsilon \to 0} \frac{h_{\varepsilon}(z)}{\log(1/\varepsilon)} = \alpha$$

where $h_{\epsilon}(z)$ represents the circle average of the field around z at radius ε .

Hu-Miller-Peres'10

Let T_{α} denote the set of α -thick points. Then almost surely, the Hausdorff dimension dim_H(T_{α}) satisfies

$$\dim_{H}(\mathsf{T}_{\alpha}) = 2 - \frac{\alpha^{2}}{2}, \alpha \leq 2$$

and T_{α} is almost surely empty if $\alpha > 2$.

Our results

We aim to further investigate the multifractal property of the limit set Λ_r .

We should first start from the simplest setting to see what we can get. Then try to apply our argument to the general case.

In this talk, **we focus exclusively on BRWs on free groups**, which are the simplest hyperbolic group.

Rate of escape

Hutchcroft'20: Almost surely, for any $\omega \in \Lambda_r$, there is unique genealogical line $t = (t_n)_n \in \partial \mathcal{T}$ such that $\operatorname{dist}_{\mathbb{F}}(V(t_n), x) = 0$.

- We use the rate of escape of the walk (V(t_n))_{n≥1} to describe the degree of singularity around the point ω = V(t) in the fractal Λ_r.
- For each $\alpha \in [0, 1]$, define

$$\Lambda_r(\alpha) := \left\{ \omega \in \partial \mathbb{F} : \exists t \in \partial \mathcal{T}, V(t) = \omega \text{ s.t. } \lim_{n \to \infty} \frac{|V(t_n)|}{n} = \alpha \right\}.$$

Then

$$\Lambda_r = \Lambda_r^{\text{no limits}} \cup \bigcup_{\alpha \in [0,1]} \Lambda_r(\alpha).$$

Question: Hausdorff dimension of $\Lambda_r(\alpha)$?

Rate function

- Let L^{*} denote the rate function of the large deviation principle for the sequence (|Z_n|/n)_{n≥1}, where (Z_n)_{n≥1} denotes the RW with step distribution μ on F.
- L^* is convex on [0, 1] and attains its minimum $L^*(c_{RW}) = 0$ at

$$C_{\mathrm{RW}} := \lim_{n \to \infty} \frac{|Z_n|}{n}.$$

• $L^*(0) = \ln R$.



Our results

Lai-M.-Wang'24+

Let $r \in (1, R]$. Almost surely, for any $\alpha \in [0, 1]$, $\Lambda_r(\alpha)$ is nonempty if and only if $L^*(\alpha) \leq \ln r$, and

$$\dim_{\mathrm{H}} \Lambda_{r}(\alpha) = \frac{\ln r - L^{*}(\alpha)}{\alpha}.$$

Here, $\alpha = 0$ is permissible only when r = R, in which case $\frac{\ln R - L^*(0)}{0}$ should be interpreted as $\lim_{\alpha \downarrow 0} \frac{L^*(0) - L^*(\alpha)}{\alpha} = -(L^*)'(0)$.



An interesting corollary

Lai-M.-Wang'24+

Let $r \in (1, R]$. There exists unique $\alpha(r) \in [0, C_{RW})$ such that

 $\dim_{\mathrm{H}} \Lambda_r = \dim_{\mathrm{H}} \Lambda_r(\alpha(r)).$



Figure 2: Illustration for $\alpha(r)$ in subcritical case 1 < r < R (left) and critical case r = R (right).

For $0 \le \alpha \le \beta \le 1$, define

$$\Lambda_r(\alpha,\beta) := \left\{ \omega \in \partial \Gamma : \exists t \in \partial \mathcal{T}, V(t) = \omega \text{ s.t.} \\ \lim_{n \to \infty} \frac{|V(t_n)|}{n} = \alpha, \lim_{n \to \infty} \frac{|V(t_n)|}{n} = \beta \right\}.$$

Then we have the decomposition

$$\Lambda_r := \cup_{\alpha \leq \beta} \Lambda_r(\alpha, \beta)$$

Our second result concerns the Hausdorff dimension of each sub-fractal $\Lambda_r(\alpha, \beta)$.

Lai–M.–Wang'24+

Assume that μ is isotropic (i.e., $\mu(a) = \mu(b) = \frac{1-\mu(e)}{4}$), Let $r \in (1, R]$. Almost surely, for any $[\alpha, \beta] \subset [0, 1]$, $\Lambda_r(\alpha, \beta)$ is nonempty if and only if $[\alpha, \beta] \subset I(r)$, and

$$\dim_{\mathrm{H}} \Lambda_r(\alpha, \beta) = \min_{q \in \{\alpha, \beta\}} \frac{\ln r - \mathcal{L}^*(q)}{q}$$

For **anisotropic** step distribution μ (that is $\mu(a) \neq \mu(b)$), we only obtained partial results:

Lai–M.–Wang'24+

Let $r \in (1, R]$. Almost surely, for any $[\alpha, \beta] \subset [0, 1]$, $\Lambda_r(\alpha, \beta)$ is nonempty if and only if $[\alpha, \beta] \subset I(r)$, and in this case the Hausdorff dimension of $\Lambda_r(\alpha, \beta)$ satisfies

$$\min_{q \in \{\alpha,\beta\}} \frac{\ln r - L^*(q)}{q} \leq \dim_{\mathrm{H}} \Lambda_r(\alpha,\beta) \leq \frac{\ln r - L^*(\alpha)}{\alpha}$$

Sketch of the proof

The energy method

Let $\theta \ge 0$ and ν be a mass distribution on a metric space (X, d). Define the θ -potential with respect to ν as

$$\mathbf{I}(\theta,\nu) := \int_X \frac{1}{d(x,y)^{\theta}} \nu(\mathrm{d} x) \nu(\mathrm{d} y).$$

If there exists $I(\theta, \nu) < \infty$, then we have $\dim_{\mathrm{H}} X \ge \theta$.

Lower bound: the energy method

Conditionally on the BRW, we select $x_n \in \mathbb{F}, v_n \in \mathcal{T}$ recursively.

- Set x_0 the identity of \mathbb{F} and v_0 be the root of \mathcal{T} .
- Given $x_j, v_j, 1 \le j \le n-1$, we choose x_n uniformly at random from the set

$$\mathbb{L}_{n} := \left\{ x \in \mathbb{F} : |x| = \sum_{j=1}^{n} m_{j}, \mathbf{x}_{n-1} \prec_{\mathbb{F}} x, \\ \exists u \succ \mathbf{v}_{n-1}, |u| = |\mathbf{v}_{n-1}| + \lfloor m_{n}/\eta_{n} \rfloor, V(u) = x \right\}$$

• Above, the sequences $(m_n) \subset \mathbb{N}$ and (η_n) are carefully chosen so that $\mathbf{x}_{\infty} := \lim_{n \to \infty} \mathbf{x}_n \in \partial \mathbb{F}$ belongs to $\Lambda_r(\alpha)$.

Let \mathbb{Q}_{α} denote the distribution of x_{∞} given the BRW. By some computations, to employ the energy method, it suffices to study

$$\mathsf{I}(\theta;\widehat{\mathbb{Q}}_{\alpha},\mathbf{x}_{\infty})\approx\sum_{n}e^{(\theta-\delta)M_{n}}\prod_{j=1}^{n}\frac{1}{\#\mathbb{L}_{n}}$$

Lower bound: the energy method

• The difficulty in proving that \mathbb{Q}_{α} has **finite energy** is to show that

$$N_{n,q}^{\mathbb{F}} := \#\{u \in \mathcal{T}, |u| = n, V(u) = qn\} \text{ and } N_{n,q}^{\mathbb{F}} := \#\{x \in \mathbb{F}^d, |x| = n, \exists |u| = n/q, V(u) = x\}$$

concentrate around their mean, respectively.

 We prove an exponential decay for sample paths LDP of the RW (we do NOT get the rate function)

$$\mathbb{P}(\exists k \leq n, |Z_k - qk| > \delta n \mid |Z_n| = qn) \lesssim e^{-C(\delta)n}$$

Then we can consider the **truncated** version of $N_{n,q}^{\mathcal{T}}$:

$$\widetilde{N}_{n,q,\delta}^{\mathcal{T}} := \#\{|u| = n : V(u) = qn, |V(u_k) - qk| \le \delta n, \forall k \le n\}$$

We employ truncated second moment method and bootstrap argument, and finally get $\mathbb{P}(N_{n,q}^{\mathcal{T}} \leq [\mathbb{E}N_{n,q}^{\mathcal{T}}]^{1-\epsilon}) \lesssim e^{-\sqrt{n}}$.

• Using an inequality for inhomogeneous GW process in Aidekon-Hu-Shi'19 and enhencing the truncation, we get $\mathbb{P}(N_{n,q}^{\mathbb{F}} \leq [\mathbb{E}N_{n,q}^{\mathbb{F}}]^{1-\epsilon}) \lesssim e^{-\sqrt{n}}$.

Upper Bound: covering

$$\Lambda_r(\alpha,\beta) \subset \bigcap_{\epsilon>0} \bigcap_{k\geq 1} \bigcup_{m\geq k} \bigcup_{\alpha-\epsilon<\frac{m}{n}<\alpha+\epsilon} \bigcup_{\substack{x\in\mathbb{F}_m\\ \exists u\in\mathcal{T}_n, V(u)=x}} \{\omega\in\partial\mathbb{F}: \omega_m=x\}.$$



Upper Bound: covering

Thus

$$\begin{aligned} \mathcal{H}_{e^{-k}}^{s}\left(\Lambda_{r}(\alpha)\right) &\leq \sum_{m\geq k} e^{-sm} \sum_{\alpha-\epsilon < \frac{m}{n} < \alpha+\epsilon} \sum_{x\in\mathbb{F}_{m}} \mathbf{1}_{\{\exists u\in\mathcal{T}_{n} \text{ s.t. } V(u)=x\}} \\ &\leq \sum_{m\geq k} e^{-sm} \sum_{\alpha-\epsilon < \frac{m}{n} < \alpha+\epsilon} \sum_{u\in\mathcal{T}_{n}} \mathbf{1}_{\{|V(u)|=m\}}. \end{aligned}$$

By applying the many-to-one formula,

For $s > \frac{\ln r - L^*(\alpha)}{\alpha} + \delta$, the series is finite, letting $k \to \infty$ we obtain $\mathcal{H}^s(\Lambda_r(\alpha)) = 0$ a.s. In order words, $\dim_{\mathrm{H}}(\Lambda_r(\alpha)) \leq \frac{\ln r - L^*(\alpha)}{\alpha}$. \Box

When $\alpha = 0$ a careful analysis is needed.

$\Lambda_r(\alpha,\beta)$ with $\alpha < \beta$

We believe that the following refined covering captures the correct dimension

$$\Lambda_r(\alpha,\beta) \subset \bigcap_{q \in [\alpha,\beta]} \bigcup \left\{ V(\partial \mathcal{T}(u)) : u \in \mathcal{T}, |V(u)|/|u| \in [q-\epsilon,q+\epsilon] \right\}.$$

We reduce the problem to verify an inequality

$$rac{\hat{P}(s)}{P'(s)} - s + \ln R \leq 0 ext{ for all } s \in (-\infty, \ln R).$$

- I the isotropic setting: the difference between P(s) and P(s) is a constant, and hence P'(s) = P'(s). Since P(s) is convex and P(ln R) = 0, the inequality follows immediately.
- In anisotropic case, it seems hard (at least for us) to verify this.

Thanks!