On Atypical Behaviors of Martingale Limits and Level Sets in Branching Random Walks

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Outline

Branching random walk

BRW conditioned on large martingale limits

Level sets of branching Brownian motion

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Branching random walk: $(V(u), u \in \mathbb{T})$

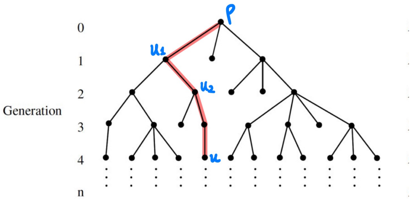

Figure: Galton-Watson tree $\mathbb{T}$

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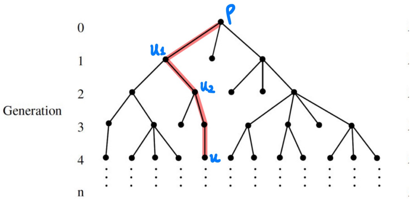

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- Given a *supercritical* GW tree $\mathbb{T}$ with root ρ, $(A_e, e \in E(\mathbb{T}))$ are i.i.d. r.v.'s
- $[\rho, u]$ is the geodesic on the tree $\mathbb{T}$ connecting ρ and $u \in \mathbb{T}$.
- Let $V(u) := \sum_{e \in [\rho, u]} A_e$.
- $(V(u), u \in \mathbb{T})$ is the branching random walk.

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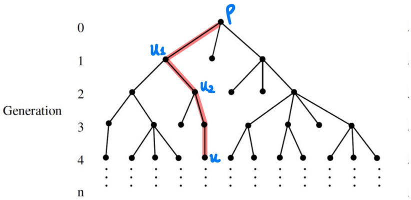

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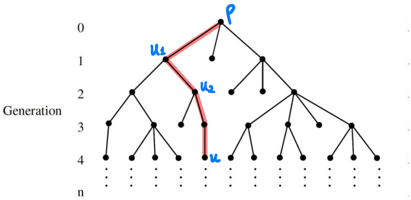

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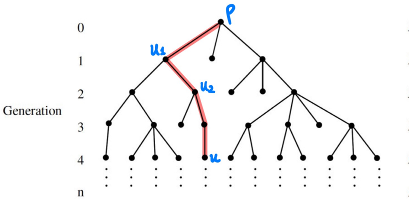

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- This is also the *Gaussian free field* on the binary tree.
- A useful function, log-Laplace transform:

$$\Psi(\theta) := \log \mathbb{E}\left[\sum_{|u|=1} e^{-\theta V(u)}\right] \in (-\infty, +\infty], \Psi(0) > 0$$

In binary Gaussian case: $\Psi(\theta) = \log 2 + \frac{\theta^2}{2}$.

Additive martingale

- $\mathbb{E}\left[\sum_{|u|=1} e^{-\theta V(u)}\right] = e^{\Psi(\theta)}$. binary Gaussian case: $\Psi(\theta) = \log 2 + \frac{\theta^2}{2}$.
- Define the additive martingale

$$W_n(\theta) = \sum_{|u|=n} e^{-\theta V(u) - n\Psi(\theta)}.$$

$W_n(\theta)$ is a non-negative martingale, hence has a a.s. limit $W_\infty(\theta)$.

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- Directed polymers on a disordered tree.
- Intermediate level set of BRW. Let $Z_n(A) = \sum_{|u|=n} 1_{V(u)\in A}$. Under mild conditions of the BRW, with probability 1, for $0 < x <$ speed of BRW

$$\frac{Z_n[xn, \infty)}{\mathbb{E}Z_n[xn, \infty)} \to W_\infty(x^*)$$

where x^* is the point such that $\Psi'(x^*) = x$.

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  converges a.s. to $W_\infty(\theta) \geq 0$.
- (Biggins'77) showed that

$$\mathbb{P}(W_\infty(\theta) > 0) > 0$$

  if and only if

$$\theta\Psi'(\theta) < \Psi(\theta),$$

  and $\mathbb{E}[W_1(\theta) \log_+ W_1(\theta)] < \infty$.
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Derivative martingale

- $\frac{d}{d\theta} W_n(\theta) = D_n(\theta)$

$$= -\sum_{|u|=n} (V(u) + n\Psi'(\theta)) e^{-\theta V(u) - n\Psi(\theta)}$$

- Signed martingale with $\mathbb{E}[D_n(\theta)] = 0$.
- Convergence criterion?

W.L.O.G., take $\theta = 1$, assume

$$\Psi'(1) < \Psi(1) = 0.$$

Why derivative martingale

Lacoin-Rhodes-Vargas'22:

- A goal from physics which I don't understand at all:

  to construct the path integral

$$\langle F \rangle_{\mathrm{ML},g} = \int F(\varphi) e^{-\beta \mathcal{S}_{\mathrm{M}}(\varphi,g) - \mathcal{S}_{\mathrm{L}}(\varphi,g)} \mathcal{D}\varphi. \qquad (1.11)$$

  Compared to the Liouville path integral which corresponds to $\beta = 0$, there is a substantial additional difficulty in defining (1.11) due to the potential term $(\gamma\varphi)e^{\gamma\varphi}$ in (1.10). Making sense of (1.11) requires controlling this term from below, a nontrivial

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- Actions from physics which I don't understand at all:

$$\mathcal{S}_{\mathrm{L}}(\varphi, g) := \frac{1}{4\pi} \int_M \left(|d\varphi|_g^2 + QK_g\varphi + 4\pi\mu e^{\gamma\varphi} \right) \mathrm{dv}_g, \tag{1.5}$$

$$\mathcal{S}_{\mathrm{M}}(\varphi, g) = \int_M \Big(2\pi(1-\mathbf{h})\phi\Delta_g\phi + \Big(\frac{8\pi(1-\mathbf{h})}{V_g} - K_g\Big)\phi + \frac{2}{1-\frac{\gamma^2}{4}} \frac{1}{V_{\hat{g}}} (\gamma\varphi)e^{\gamma\varphi} \Big) \mathrm{dv}_g, \tag{1.10}$$

Why derivative martingale

- Interpretations:

in Section 3, but let us just mention that the construction is based on interpreting $e^{-\frac{1}{4\pi}\int_M |d\varphi|_g^2 \, dv_g} \mathcal{D}\varphi$ as a GFF measure and expressing the other terms in the actions as functions of the GFF. With this in mind, the term $e^{\gamma\varphi}$ in the Liouville action (1.5) gives rise to GMC and the $(\gamma\varphi)e^{\gamma\varphi}$ term in the Mabuchi action (1.10) gives rise to a derivative (with respect to γ) of GMC.

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tity (1.13) is well defined and nontrivial for $\gamma \in (0,2)$. Now consider what we call *derivative GMC* (DGMC for short)[11]

$$M'_\gamma(dx) := \big(X(x) - \gamma\mathbb{E}[X^2(x)]\big)e^{\gamma X(x) - \frac{\gamma^2}{2}\mathbb{E}[X^2(x)]}\,\mathrm{v}_g(\mathrm{d}x) \qquad (1.14)$$

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L^p convergence of martingales

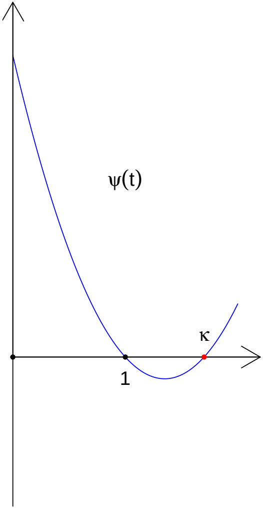

Figure: $\Psi(1) = 0 > \Psi'(1)$

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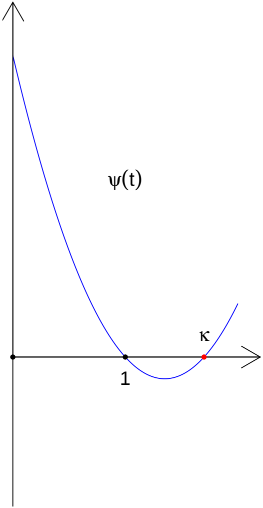

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Assume that there is $\kappa \in (1, \infty)$ s.t. $\Psi(\kappa) = 0$, and $\exists \delta > 0$ s.t.

$$\mathbb{E}\left[\left(\sum_{|u|=1}(1 + |V(u)|)e^{-V(u)}\right)^{\kappa+\delta}\right] < \infty.$$

Then, $\forall p \in (0, \kappa)$,

$$W_n \xrightarrow{a.s., L^p} W_\infty$$

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binary Gaussian case: $\kappa = 2log2/\theta^2$.

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Tails of W_∞ and D_∞

- (Liu'2000) showed that there is constant $C_W > 0$ satisfying

  $$\mathbb{P}(W_\infty > x) \sim C_W\, x^{-\kappa}$$

  that as $x \to \infty$, by using that W_∞ satisfying some random difference equation

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$$\mathbb{P}(D_\infty(\theta) < -x) = e^{-\Theta(1)x^\gamma}$$

with $\gamma = \frac{2\log 2}{\theta^2}$. Partially confirmed by Bonnefont-Vargas for small θ.

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Inspired by Madaule's method,

Theorem(Chen-de Raphélis-M. 24+)
As $x \to \infty$, conditioned on $\{W_\infty \geq x\}$,

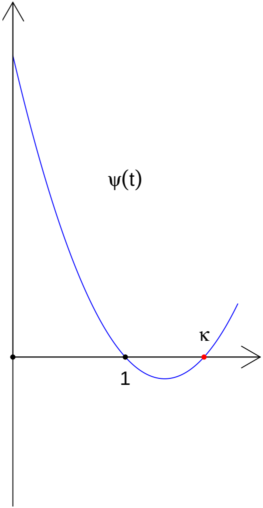
$$\frac{D_\infty}{x \log x} - \left[\frac{\Psi'(\kappa) - \Psi'(1)}{\Psi'(\kappa)}\right] \frac{W_\infty}{x} \xrightarrow{\mathbb{P}} 0$$

and

$$\mathbb{P}(D_\infty > x) \sim C_D \frac{(\log x)^\kappa}{x^\kappa}$$

with $C_D = C_W \left(\frac{\Psi'(\kappa) - \Psi'(1)}{\Psi'(\kappa)}\right)^\kappa$.

Branching random walk in κ-case



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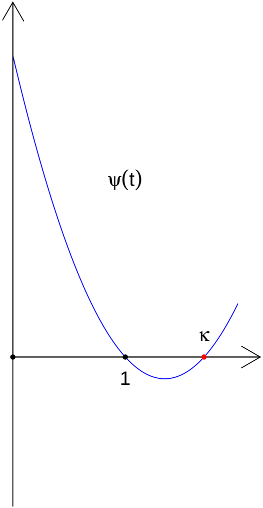

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Global minimum

- (Biggins'1976) showed

$$\frac{\inf_{|u|=n} V(u)}{n} \xrightarrow{a.s.} -\Gamma.$$

- $\Gamma := \inf_{\theta>0} \frac{\Psi(\theta)}{\theta} < 0$

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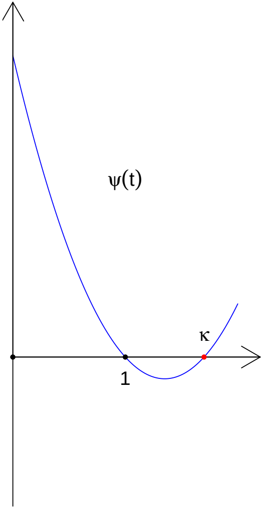

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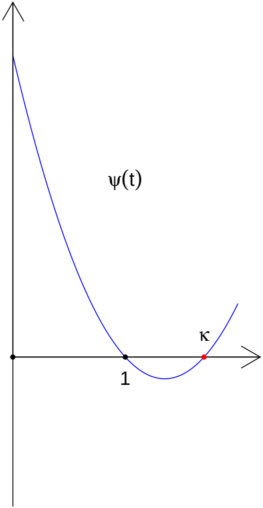

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$$\mathtt{M} := \inf_{u\in\mathbb{T}} V(u) \in \mathbb{R}.$$

Observe $\mathtt{M} = V(u^*)$,

$$W_\infty = \sum_{|u|=n} e^{-V(u)} W_\infty^{(u)} \geq e^{-\mathtt{M}} W_\infty^{(u^*)}$$

Conditioned on global minimum $\mathtt{M} \le -z = -\log x$

$\mathtt{M} = V(u^*)$ is attained at generation $|u^*|$.

$$W_\infty \asymp e^{-\mathtt{M}} W_\infty^{(u^*)},\ D_\infty \asymp e^{-\mathtt{M}}[D_\infty^{(u^*)} + (-V(u^*) - |u^*|\Psi'(1))W_\infty^{(u^*)}]$$

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Theorem[Chen–de Raphélis–M.''24+]

Under suitable moment condition, as $z \to \infty$,

$$\mathbb{P}\left((\sum_{u\in\mathbb{T}} \delta_{V(u)-\mathtt{M}}, \sqrt{\frac{\Psi'(\kappa)}{z}}(|u^*| - \frac{z}{\Psi'(\kappa)}), e^{\mathtt{M}} W_\infty, \frac{e^{\mathtt{M}} D_\infty}{|u^*|}, \mathtt{M} + z) \in \cdot \,\middle|\, \mathtt{M} \le -z \right)$$
$$\to \mathbb{P}((\mathcal{E}_\infty, G, (\Psi'(\kappa) - \Psi'(1))Z, Z, -U) \in \cdot)$$

where $(\mathcal{E}_\infty, Z)$, U and G are independent, $G \sim N(0, \frac{\Psi''(\kappa)}{\Psi'(\kappa)^2})$, $U \sim \mathrm{Exp}(\kappa)$
And $\mathbb{E}[Z^\kappa] < \infty$.

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$$\to \mathbb{P}\left((\widehat{\mathcal{E}}_\infty, e^U \frac{\Psi'(\kappa) - \Psi'(1)}{\Psi'(\kappa)}, e^U, \log \hat{Z} - U) \in \cdot \right)$$

where $\mathbb{E}[f(\widehat{\mathcal{E}}_\infty, \hat{Z})] = \frac{1}{\mathbb{E}[Z^\kappa]} \mathbb{E}[Z^\kappa f(\mathcal{E}_\infty, Z)]$.

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$$\mathbb{P}\left((\sum_{u \in \mathbb{T}} \delta_{V(u)-\mathtt{M}}, \frac{D_\infty}{x \log x}, \frac{W_\infty}{x}, \mathtt{M} + \log x) \in \cdot \middle| W_\infty \geq x \right)$$

$$\to \mathbb{P}\left((\widehat{\mathcal{E}}_\infty, e^U \frac{\Psi'(\kappa) - \Psi'(1)}{\Psi'(\kappa)}, e^U, \log \hat{Z} - U) \in \cdot \right)$$

where $\mathbb{E}[f(\widehat{\mathcal{E}}_\infty, \hat{Z})] = \frac{1}{\mathbb{E}[Z^\kappa]} \mathbb{E}[Z^\kappa f(\mathcal{E}_\infty, Z)]$.

- Conditioned on $\{W_\infty \geq x\}$,

$$\frac{D_\infty}{x \log x} - [\frac{\Psi'(\kappa) - \Psi'(1)}{\Psi'(\kappa)}] \frac{W_\infty}{x} \xrightarrow{\mathbb{P}} 0.$$

- So, $\mathbb{P}(D_\infty \geq x \log x) \asymp \mathbb{P}(W_\infty \geq x) \sim C_0 x^{-\kappa}$.

An key estimate of high moments of additive martingale

Let $\mathtt{M}_n = \inf_{|u|\leq n} V(u)$.

Lemma [Chen-de Raphélis-M. 24+]

For $\delta \in (0,1)$ small we have

$$\mathtt{E}\left[W_n^{\kappa+\delta} 1_{\{\mathtt{M}_n \geq -x\}}\right] \leq Ce^{\delta x} \quad \forall n \in \mathbb{N} \cup \{\infty\}, x \geq 0,$$

and

$$\mathtt{E}\left[|D_n|^{\kappa+\delta}\, 1_{\{\mathtt{M}_n \geq -x\}}\right] \leq C' e^{\delta x}(1+x)^{\kappa+\delta} \quad \forall n \in \mathbb{N} \cup \{\infty\}, x \geq 0.$$

Outline

Branching random walk

BRW conditioned on large martingale limits

Level sets of branching Brownian motion

Branching Brownian motion

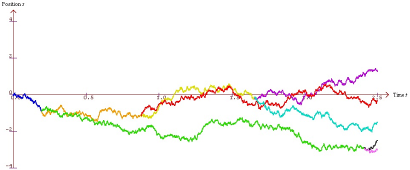
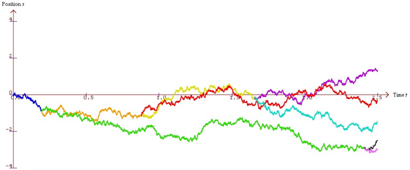


Figure: BBM

Life time = Exp(1)
Motion = standard Brownian motion
At time t, $\{\Phi_k(t)\}_{1 \le k \le N_t}$=positions
log-laplace transform: $\Psi(\theta) = 1 + \frac{\theta^2}{2}$.

Branching Brownian motion



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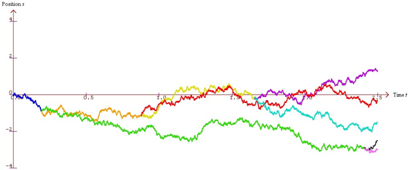

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(Liu'2000)

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with $\kappa_\theta = 2/\theta^2$.

Level sets in branching Brownian motion

Positions at time t, $\{\Phi_k(t)\}_{1\leq k\leq N_t}$

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Level sets For $A \subset \mathbb{R}$,

$$Z_t(A) := \sum_{1\le k\le N_t} \mathbf{1}_{\{\Phi_k(t)\in A\}}$$

- $\mathbb{E}[Z_t[xt,\infty)] \sim \frac{1}{\sqrt{2\pi t x}} e^{t(1-\frac{x^2}{2})}$

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$$\frac{Z_t[xt, xt + y]}{\mathbb{E}[Z_t[xt, \infty)]} \xrightarrow{a.s.} W_\infty(x)(1 - e^{-xy}).$$

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- (Glenz-Kistler-Schmidt'18)

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Large deviation of level sets

Typically, $Z_t[xt, \infty) \approx \frac{e^{(1-\frac{x^2}{2})t}}{\sqrt{2\pi t x}} W_\infty(x)$.

Theorem(Aïdékon-Hu-Shi' 2019)

For $x > 0$ and $(1 - \frac{x^2}{2})_+ < a < 1$, let $I(a, x) = \frac{x^2}{2(1-a)} - 1$,

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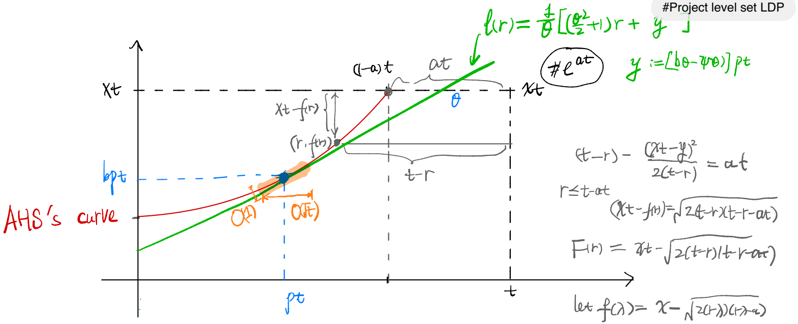
#Project level set LDP



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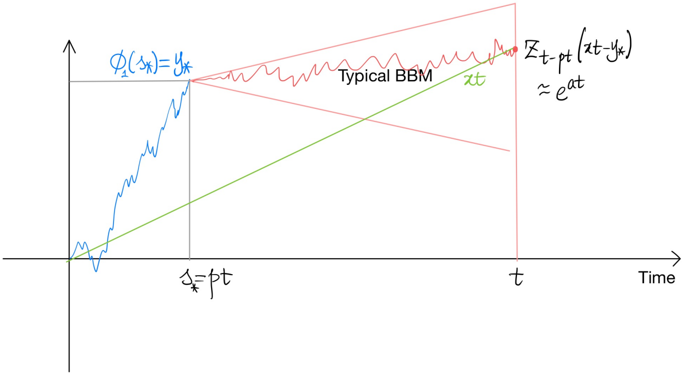

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3.1. Lower bound. The strategy of the lower bound in Theorem 1.1 is as follows. Let $\varepsilon > 0$. Let $s_* = \frac{(1-a)[x^2-2(1-a)]}{x^2-2(1-a)^2}t$ and $y_* = \frac{x}{1-a}s_*$ be the maximizer in (3.1) of Lemma 3.1. Let the BBM reach $[y_*, \infty)$ at time s_* (which, by (1.1), happens with probability at least $\exp[-(1+\varepsilon)(\frac{y_*^2}{2s_*} - s_*)] = \mathrm{e}^{-(1+\varepsilon)I(a,x)t}$ for all sufficiently large t); then, after time s_*, the system behaves "normally" in the sense that by (1.2), with probability at least $1-\varepsilon$ for all sufficiently large t, the number of descendants positioned in $[xt,\infty)$ at time t of the particle positioned in $[y_*,\infty)$ at time s_* is at least $\exp\{(1-\varepsilon)[(t-s_*) - \frac{(xt-y_*)^2}{2(t-s_*)}]\}$ (which is $\mathrm{e}^{(1-\varepsilon)at}$); note that the condition $0 < \frac{xt-y_*}{t-s_*} < 2^{1/2}$ in (1.2) is automatically satisfied. Consequently, for all sufficiently large t,

$$\mathbb{P}\Big(N(t,x) \geq \mathrm{e}^{(1-\varepsilon)at}\Big) \geq (1-\varepsilon)\,\mathrm{e}^{-(1+\varepsilon)I(a,x)t}.$$

Since $\varepsilon > 0$ can be as small as possible, this yields the lower bound in Theorem 1.1. □

Figure: Lower bound by 10 lines ; Upper bound by 2+4 pages

Large deviation of level sets: upper bound

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A Short Proof using martingale tail inequality.

$W_\infty(\theta) = \sum_{k=1}^{N_t} e^{\theta\Phi_k(t)-t\Psi(\theta)} W_\infty^{(k)}$ with $W_\infty^{(k)}$ i.i.d. copies of $W_\infty(\theta)$.

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$$\begin{aligned}
\mathbb{P}(Z_t[xt,\infty) \geq e^{at}) &\leq \mathbb{P}\left(W_\infty(\theta) \geq e^{\theta xt - t\Psi(\theta)} \sum_{1\leq k\leq e^{at}} W_\infty^{(k)}\right) \\
&\leq \mathbb{P}\left(W_\infty(\theta) \geq e^{[\theta x-\Psi(\theta)]t}\frac{1}{2}e^{at}\right) + \mathbb{P}\left(\sum_{1\leq k\leq e^{at}} W_\infty^{(k)} < \frac{1}{2}e^{at}\right) \\
&\lesssim \quad e^{-\kappa_\theta[\theta x-\Psi(\theta)+a]t} \quad + \quad \underbrace{e^{-\epsilon e^{at}}}_{Chernorff's-bound}
\end{aligned}$$

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Take optimal $\theta = \frac{2(1-a)}{x} \in (0,\sqrt{2})$, then $\kappa_\theta[\theta x - \Psi(\theta)+a] = I(a,x)$.

Precise large deviation for level sets

AHS'19: $\mathbb{P}(Z_t[xt, \infty) \geq e^{at}) = e^{-I(a,x)t+o(t)}$ where $I(a,x) = \frac{x^2}{2(1-a)} - 1$.

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Theorem [Chen–M. '24+]

For $x > 0$ and $(1 - \frac{x^2}{2})_+ < a < 1$,

$$\mathbb{P}(Z_t[xt,\infty) \geq e^{at}) \sim C_{x,a} t^{-\kappa_\theta/2} e^{-I(a,x)t}.$$

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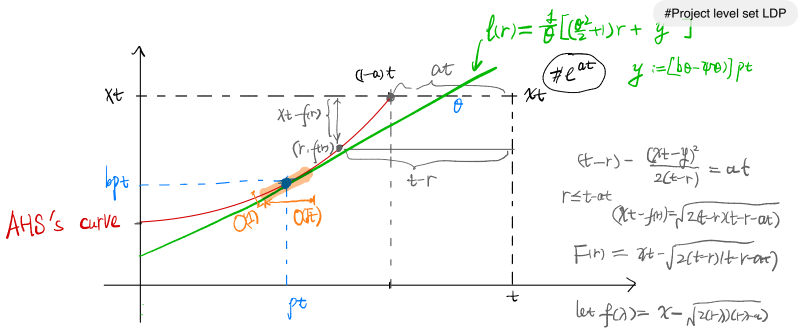
Moreover, for $y > 0$

$$\mathbb{P}(Z_t[xt,\infty) \geq \frac{y}{\sqrt{t}} e^{at}) \sim C_{x,a} y^{-\kappa_\theta} e^{-I(a,x)t}.$$

A road to up-to-constant estimate



Observation: BBM hit the green line iff

$$\inf_{s>0} \min_{u\in N_s} (\frac{\theta^2}{2} + 1)s - \Phi_s(u) \leq -\left(1 - \frac{\theta^2}{2}\right) pt$$

A road to up-to-constant estimate

- Decomposition

$$Z_t[xt, +\infty) = \sum_{u \in N_s} Z^{(u)}_{t-s}(xt - \Phi_s(u)) \quad \forall s > 0$$

are independent given $\mathcal{F}_s = \sigma(\Phi_r(v) : r < s, v \in N_r)$

- A inequality: (X_i) independent. $X_i \geq 0$.

$$\mathbb{P}\left(\sum X_i > t\right) \leq \underbrace{\sum \mathbb{P}\left(X_i > \frac{t}{\lambda}\right)}_{①} + \underbrace{\left(\frac{e \sum \mathbb{E}X_i}{t}\right)^{\lambda}}_{②}$$

- $$\mathbb{P}\left(Z_t[xt, +\infty) \geq \frac{1}{\sqrt{t}} e^{at}, \ \inf_{s > 0} \left(\frac{\theta^2}{2}+1\right)s - \theta \Phi_s(u) \geq -\left(1 - \frac{\theta^2}{2}\right)pt + z \,\middle|\, \mathcal{F}_{pt} \right)$$

Comes from BBM hit the line

$$\leq ① + \underbrace{e^{\lambda} \cdot \left(\frac{\mathbb{E} \sum_{u \in N_{pt}} \left[Z^{(u)}_{(1-p)t}(xt - \Phi_{pt}(u)) \middle| \mathcal{F}_{pt} \right]}{e^{at}/\sqrt{t}} \right)^{\lambda} \mathbb{1}_{\left\{ \inf_{r < pt} \left(\frac{\theta^2}{2}+1\right)r - \theta \Phi_r(u) \geq \cdots \right\}}}_{②}$$

A road to up-to-constant estimate

- $\mathbb{P}\left(Z_t[x_t,+\infty) \geq \frac{1}{\sqrt{t}} e^{at},\ \inf_{s\geq 0} (\frac{\theta^2}{2}+1)s - \theta\Phi_s(u) \geq -(1-\frac{\theta^2}{2})pt + z \,\Big|\, \mathcal{F}_{pt}\right)$

$$\leq (1) + e^{\lambda} \cdot \left(\frac{\mathbb{E}\sum_{u\in N_{pt}} \left[Z^{(u)}_{(1-p)t}(x_t - \Phi_{pt}(u)) \,|\, \mathcal{F}_s\right]}{e^{at}/\sqrt{t}}\right)^{\lambda} 1_{\left\{\inf_{r\leq pt} (\frac{\theta^2}{2}+1) r - \theta\Phi_r(u) \geq \cdots\right\}}$$

$$\lesssim \sum_{u\in N_{pt}} e^{\theta \Phi_{pt}(u) - 2pt}$$

$\frac{2}{\theta^2}(1-\frac{\theta^2}{2})p = I(a,x)$

- Chose $\lambda = \frac{2}{\theta^2}+\delta$ with $\delta\in(0,1)$ e.g. $\delta = 1/2$.

$$\text{Prob} \leq e^{-(\frac{2}{\theta^2}+\delta)(1-\frac{\theta^2}{2})pt}\, \mathbb{E}\left(W_{pt}(\theta)^{\frac{2}{\theta^2}+\delta} 1_{\left\{\inf_{r\leq pt}(\frac{\theta^2}{2}+1)r - \theta\Phi_r(u) \geq -(1-\frac{\theta^2}{2})pt + z\right\}}\right)$$

$$\lesssim e^{-(\frac{2}{\theta^2}+\delta)(1-\frac{\theta^2}{2})pt}\, e^{\delta(1-\frac{\theta^2}{2})pt - \delta z} = e^{-I(a,x)t - \delta z}$$

Conditioned overlap distribution

- Given the BBM up to time t, we choose two individuals u^1, u^2 independently and uniformly from x-level set $\{v \in \mathcal{N}_t : \Phi_v(t) \geq xt\}$.

Theorem [Chen–M. '24+]

We have the following conditional central limit theorem:

$$\left(\frac{|u^1 \wedge u^2| - pt}{c\sqrt{pt}}, \frac{\Phi_{u^1}(|u^1 \wedge u^2|) - bpt}{c'\sqrt{t}} \mid Z_t[xt, \infty) \geq \frac{1}{\sqrt{t}} e^{at} \right) \Rightarrow (G, G)$$

where G is a standard Gaussian random variable.

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where G is a standard Gaussian random variable.

As a comparison, without conditioned on large level set size,

$$\left(|u^1 \wedge u^2|, X_{|u^1 \wedge u^2|}(u^1) \right) \text{ converges in law.}$$

Conditioned maximum

- Let $M_t := \max_{u \in \mathcal{N}_t} \Phi_u(t)$ be the maximum position.

Theorem [Chen–M. '24+]

Set
$$v := bp + \sqrt{2}(1-p) > \sqrt{2}$$
then
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where G is a standard Gaussian random variable.

As a comparison, without conditioned on large level set size,

$$M_t - \sqrt{2}t + \frac{3}{2\sqrt{2}} \log t \text{ converges in law.}$$

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Thank you!