Large deviations of level sets and martingale limits for branching Brownian motion

Heng Ma

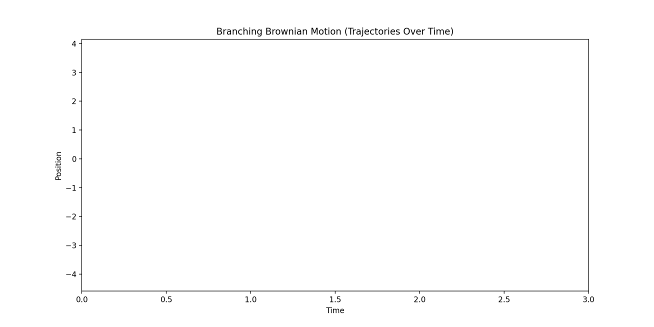
Feb 19, 2025

Peking University

Brownian Brownian Motion

A binary branching Brownian motion $\{(X_s(u) : s \leq t) : u \in \mathcal{N}_t\}$:

- Start with one particle undergoing standard Brownian motion. After an independent $\mathrm{Exponential}(1)$ time, it splits into **two** offspring.
- Offspring begin independent Brownian motions from the splitting point and repeat the process independently.



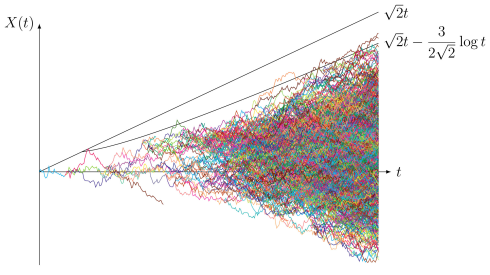
BBM viewed as Correlated Gaussian processes

Maximum of Branching Brownian motion

Bramson'83: The maximum $M_t := \max_{u\in\mathcal{N}_t} X_t(u)$ satisfies

$$M_t - (\sqrt{2}t - \frac{3}{2\sqrt{2}}\ln t) \xrightarrow[t\to\infty]{\text{in law}} \text{ randomly shifted Gumbel r.v.}$$

- Comparing with i.i.d. case: $M_t^{iid} - (\sqrt{2}t - \frac{1}{2\sqrt{2}}\ln t) \xrightarrow[t\to\infty]{\text{in law}}$ Gumbel.

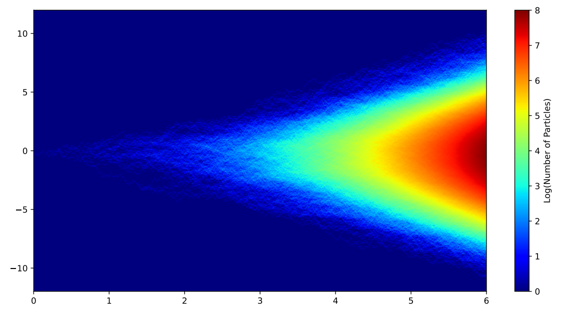


Universality: 2-dim Gaussian free field, Cover time of planar random walks, maximum of local times of random walks, ...

Level sets

For $y \in \mathbb{R}$, define the size of y-level set at time t as

$$L_t[y, \infty) := \sum_{u \in \mathcal{N}_t} 1(X_t(u) \geq y).$$



- The result of Bramson implies that the $\sqrt{2}t$-level set is empty for large t.
- An appropriate scale we should look at is xt-level set for $x \in [0, \sqrt{2})$

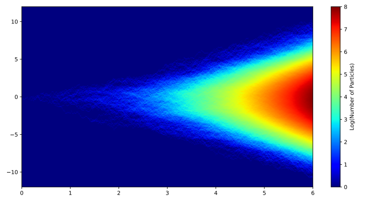
Law of Large numbers for Level sets

Biggins'79; Glenz, Kistler, Schmidt'18: For the intermediate high level set $x \in (0, \sqrt{2})$, the law of large numbers holds:

$$\frac{L_t[xt, \infty)}{\mathbb{E}L_t[xt, \infty)} \xrightarrow[t\to\infty]{a.s.} W_\infty(x)$$

where $\mathbb{E}L_t[xt, \infty) \sim \frac{1}{x\sqrt{2\pi t}} e^{(1-\frac{x^2}{2})t}$, and $W_\infty(x) > 0$ is some r.v.

- Comparing with i.i.d. case: $\frac{L_t^{iid}[xt,\infty)}{\mathbb{E}L_t[xt,\infty)} \xrightarrow[t\to\infty]{a.s.} 1$.



Biggins'79: $\frac{1}{\mathbb{E}L_t[xt,\infty)} \sum_{u\in\mathcal{N}_t} f(X_t(u) - xt) \xrightarrow[t\to\infty]{a.s.} W_\infty(x) \int f(h)e^{-xh}x\mathrm{d}h,$

Biskup-Louidor'19: A stronger version holds for 2-dim Gaussian free field.

Questions

Reminder: For $x \in (0, \sqrt{2})$, typically $L_t[xt, \infty) \approx \frac{1}{x\sqrt{2\pi t}} e^{(1-\frac{x^2}{2})t} W_\infty(x)$.

Q1. Tail probability of $W_\infty(x)$.

- What is the decay rate of $\mathbb{P}(W_\infty(x) > y)$ as $y \to \infty$?
- What happens when conditioned on $W_\infty(x)$ being unusually large?

Q2. Upper large deviation of the level set $L_t[xt, \infty)$.

Take $x > 0$ and $(1 - \frac{x^2}{2})_+ < a < 1$.

- What is the decay rate of $\mathbb{P}(L_t[xt, \infty) \geq e^{at})$ as $t \to \infty$?
- What happens when conditioned on $L_t[xt, \infty)$ being atipically large ?

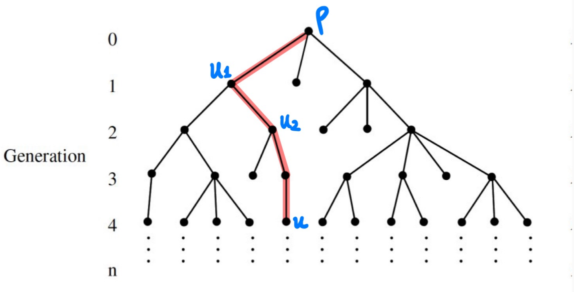
Other interesting questions: lower deviation? In particular, hard wall constriant $\mathbb{P}(L_t[-\infty, 0) = 0)$?

Martingale Limits of branching random walks

Branching random walk: $(V(u), u \in \mathcal{T})$

BRW with i.i.d. displacement:

- Sample a GW tree $\mathcal{T}$ with root ρ, $(\xi_e, e \in E(\mathcal{T}))$ are i.i.d. r.v.'s
- For each non-root vertex $u \in \mathcal{T}$, define $V(u) := \sum_{e \in \mathrm{Path}(\rho,u)} \xi_e$.



Examples:

- Gaussian free field on binary tree: $\mathcal{T}$ = binary tree and $\xi_e \sim \mathcal{N}(0,1)$.
- A slight generalization includes binary branching Brownian motion.

Additive martingales

The log-Laplace transform:

$$e^{\psi(\theta)} := \mathbb{E}\left[\sum_{|u|=1} e^{-\theta V(u)}\right] \in (0, +\infty].$$

• BBM: $\psi(\theta) := 1 + \frac{\theta^2}{2}$. • GFF on binary tree: $\psi(\theta) = \log 2 + \frac{\theta^2}{2}$.

Additive martingales:

$$W_n(\theta) := \sum_{|u|=n} e^{-\theta V(u) - \psi(\theta) n} \geq 0, \quad \mathbb{E} W_n(\theta) = 1.$$

Biggins's criterion: $\mathbb{P}(W_\infty(\theta) > 0) > 0$ iff $\theta \psi'(\theta) - \psi(\theta) < 0$.

• BBM: $|\theta| < \sqrt{2}$. • GFF on binary tree: $|\theta| < \sqrt{2 \log 2}$.

Rmk: W.L.O.G. set $\theta = 1$ and $\psi(1) = 0$. Then $W_n := \sum_{|u|=n} e^{-V(u)} \to W_\infty$.

Tail of additive martingale limit

Assumption: $\exists\, \kappa > 1$ such that $\psi(\kappa) = 0$, i.e., $\mathbb{E}\left[\sum_{|u|=1} e^{-\kappa V(u)}\right] = 1$.

(For BBM, we regard $V(u)$ as $(\frac{x^2}{2} + 1)n - x\, X_n(u)$ for $u \in \mathcal{N}_n$. Then $\kappa = \frac{2}{x^2}$.)

Liu'00: Exact decay rate of tail probability

$\mathbb{P}(W_\infty > y) \sim C\, y^{-\kappa}$, using that W_∞ satisfies some random equation $W \overset{d}{=} AW + B$ with (A, B) independent to W.

i.e., $\mathbb{P}(\ln W_\infty > y) \sim Ce^{-\kappa y}$ if you prefer exponential decay rate for a LDP.

Further Question: What happens when conditioned on the martingale limit W_∞ being unusually large? (Optimal strategy)

Martingale limit and global minimum

Further Question: What happens when conditioned on W_∞ being unusually large?

Chen-de Raphélis-M. 24+ (Optimal strategy)

The Global minimum $\mathbf{M} := \min_{u \in \mathcal{T}} V(u)$ is very negative. Precisely,

$$\mathbb{P}(W_\infty > x, |\mathbf{M} + \ln x| > z) \lesssim e^{-\delta z} x^{-\kappa}.$$

Key ingredient for the proof: higher moments estimate: for $\delta \in (0,1)$

$$\sup_{n \geq 1} \mathbb{E}\left[W_n^{\kappa+\delta}; \min_{|u| \leq n} V(u) > -y\right] \leq C_{\kappa+\delta}\, e^{\delta y}.$$

2nd ingredient: Using second moment method, we have $\mathbb{P}(\mathbf{M} \leq -y) \asymp e^{-\kappa y}$

Applying Markov inequality,

$$\mathbb{P}(W_\infty > x, \mathbf{M} \geq -\ln x + z) \lesssim \frac{e^{\delta(\ln x - z)}}{x^{\kappa+\delta}} = e^{-\delta z} x^{-\kappa}.$$

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Tail of (subcritical) derivative martingale limit

Derivative martingale:

$$D_n = \sum_{|u|=n} [V(u) + \psi'(1)n] e^{-V(u)} = \frac{\partial}{\partial \theta}\bigg|_{\theta=1} W_n(\theta)$$

Chen-de Raphélis-M. 24+: Right tail

$$\mathbb{P}(D_\infty > x) \sim C_{D_\infty} \frac{(\ln x)^\kappa}{x^\kappa}$$

Lacoin, Rhodes, Vargas'22, Conjecture: $-\ln \mathbb{P}(D_\infty < -x) \asymp x^\gamma$.

(Partial progress was made in **Bonnefont-Vargas'23**)

Conditioned BRW

Chen-de Raphélis-M. 24+: Cond. very negative minimum

$$\left(\frac{\tau_{\mathbf{M}} - \frac{x}{c_\kappa}}{\sqrt{x}}, \frac{e^{\mathbf{M}} D_\infty}{\tau_{\mathbf{M}}}, e^{\mathbf{M}} W_\infty, \sum_{u \in \mathbb{T}} \delta_{V(u) - \mathbf{M}} \mid \mathbf{M} \le -x\right)$$
$$\Longrightarrow \left(G, \left(\psi'(\kappa) - \psi'(1)\right) Z, Z, \mathcal{E}_\infty\right)$$

where G is a Gaussian r.v. independent of $(Z, \mathcal{E}_\infty)$.

Convert results above to conditioning on very large martingale limit.

Chen-de Raphélis-M. 24+: Cond. very large mart. limit

$$\left(\mathbf{M} + \ln x, \frac{W_\infty}{x}, \frac{D_\infty}{x \ln x} \mid W_\infty > x\right)$$
$$\Longrightarrow \left(\ln \widehat{Z} - \xi, e^{\xi}, \frac{\psi'(\kappa) - \psi'(1)}{\psi'(\kappa)} e^{\xi}\right)$$

where $\xi \sim \mathrm{Exp}(\kappa^{-1})$ independent of $\widehat{Z}$.

Large deviations of level sets of branching Brownian motion

Large deviation

Typically, for $x \in (0, \sqrt{2})$, $L_t[xt, \infty) \approx \frac{1}{x\sqrt{2\pi t}} W_\infty(x) e^{(1-\frac{x^2}{2})t}$.

Aïdékon-Hu-Shi'19: Rate function for LDP

For $x > 0$ and $(1 - \frac{x^2}{2})_+ < a < 1$,

$$\mathbb{P}\left(L_t[xt, \infty) \geq e^{at}\right) = e^{-I(a,x)t + o(t)},$$

where $I(a, x) := \frac{x^2}{2(1-a)} - 1$.

Can we just replace $L_t[xt, \infty)$ by $W_\infty(x) e^{(1-\frac{x^2}{2})t}$ to get the correct answer?

Short proof for upper bound

A short proof for the upper bound using martingale tail inequality.

Note that $W_{t+s}(\theta) = \sum_{u\in\mathcal{N}_t} e^{\theta X_t(u) - t\psi(\theta)} \sum_{v\in\mathcal{N}_s^{(u)}} e^{\theta(X_t(v)-X_t(u))-s\psi(\theta)}$. Letting $s\to\infty$, we get $W_\infty(\theta) = \sum_{u\in\mathcal{N}_t} e^{\theta X_t(u)-t\psi(\theta)} W_\infty^{(u)}$ with $W_\infty^{(u)} \overset{i.i.d.}{\sim} W_\infty(\theta)$.

$$\begin{aligned}
\mathbb{P}(L_t[xt,\infty) \geq e^{at}) \leq & \mathbb{P}\left(W_\infty(\theta) \geq e^{\theta xt - t\psi(\theta)} \sum_{1\leq k\leq e^{at}} W_\infty^{(k)}\right) \\
\leq & \mathbb{P}\left(W_\infty(\theta) \geq e^{[\theta x - \psi(\theta)]t} \frac{1}{2} e^{at}\right) + \mathbb{P}\left(\sum_{1\leq k\leq e^{at}} W_\infty^{(k)} < \frac{1}{2}e^{at}\right) \\
\lesssim & \underbrace{e^{-\kappa_\theta[\theta x - \psi(\theta)+a]t}}_{\text{Tail of martingale}} + \underbrace{e^{-\epsilon e^{at}}}_{Chernorff's-bound}
\end{aligned}$$

Take the optimal $\theta = \frac{2(1-a)}{x} \in (0,\sqrt{2})$. Then $\kappa_\theta[\theta x - \psi(\theta) + a] = I(a,x)$.

Precise large deviations estimates

Aïdékon-Hu-Shi'19: $\mathbb{P}\left(L_t[xt,\infty) \geq e^{at}\right) = e^{-I(a,x)t+o(t)}$ where $I(a,x) := \frac{x^2}{2(1-a)} - 1$.

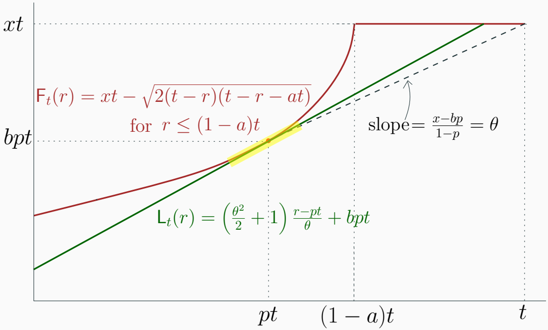
Chen-M.'24+: Precise LDP

For $x > 0$ and $(1-\frac{x^2}{2})_+ < a < 1$,

$$\mathbb{P}\left(L_t[xt,\infty) \geq \frac{y}{\sqrt{t}} e^{at}\right) \sim C_{x,a} y^{-\frac{2}{\theta^2}} e^{-I(a,x)t}.$$

Rmk: $\frac{1}{\sqrt{t}}$ is added to match the pre-factor in $\mathbb{E}[L_t[xt,\infty)] \sim \frac{1}{x\sqrt{2\pi t}} e^{(1-\frac{x^2}{2})t}$. Take $y = \sqrt{t}$ to cancel this factor.

Level set and Global minimum of global minimum of linearly transformed BBM



Observation: BBM hits the green tangent line iff

$$\mathbf{I} := \inf_{s>0} \min_{u \in N_s} \left(\frac{\theta^2}{2}+1\right) s - \theta X_s(u) \leq -\left(1 - \frac{\theta^2}{2}\right) pt$$

Why optimal strategy

Observation: BBM hits the green tangent line iff

$$\mathbf{I} := \inf_{s>0} \min_{u \in N_s} \left(\frac{\theta^2}{2}+1\right) s - \theta X_s(u) \leq -\left(1-\frac{\theta^2}{2}\right) pt$$

Chen-M.'24+ (Optimal strategy)

For any $z \geq 0$

$$\mathbb{P}(L_t[xt,\infty) \geq \frac{y}{\sqrt{t}} e^{at}, |\mathbf{I} + (1-\frac{\theta^2}{2})pt| > z) \lesssim e^{-I(a,x)t} e^{-\delta z}$$

1. **Hu-Nyrhinen'04**: $X_i \geq 0$ are independent, for any $\lambda > 0$ and $t > 0$
$$\mathbb{P}\left(\sum_i X_i > y\right) \leq \sum_{i=1} \mathbb{P}\left(X_i > \frac{y}{\lambda}\right) + \left(\frac{e\sum_i \mathbb{E}[X_i]}{y}\right)^{\lambda}.$$

2. Apply this inequality to $\mathbb{P}(L_t[xt,\infty) \geq \frac{1}{\sqrt{t}} e^{at}, \mathbf{I} \geq -\left(1-\frac{\theta^2}{2}\right) pt + z)$ and note that
$$L_t[xt,\infty) = \sum_{u \in \mathcal{N}_{pt}} L^{(u)}_{t-pt}[xt - X_{pt}(u), \infty)$$

Optimal strategy

3. We can upper bound $\mathbb{P}(L_t[xt,\infty) \geq \frac{1}{\sqrt{t}}e^{at}, \mathbf{I}_{pt} \geq -\left(1-\frac{\theta^2}{2}\right)pt+z)$ by

$$(Sum) + e^{\lambda}\left(\sum_{u\in\mathcal{N}_{pt}} \frac{\mathbb{E}\left[L^{(u)}_{t-pt}[xt - X_{pt}(u),\infty) \mid \mathcal{F}_{pt}\right]}{e^{at}/\sqrt{t}}\right)^{\lambda} 1(\mathbf{I}_{pt} \geq -(1-\frac{\theta^2}{2})pt+z)$$

4. Upper bound the blue part by

$$\sum_{u\in\mathcal{N}_{pt}} e^{-\theta X_{pt}(u)-2pt} = e^{-(1-\frac{\theta^2}{2})pt} W_{pt}(\theta)$$

5. Choose $\lambda = \kappa_\theta + \delta = \frac{2}{\theta^2} + \delta$ $(\theta := \frac{2(1-a)}{x})$ then apply higher moments estimate:

$$e^{-(1-\frac{\theta^2}{2})(\frac{2}{\theta^2}+\delta)pt}\mathbb{E}\left[W_{pt}(\theta)^{\frac{2}{\theta^2}+\delta} 1(\mathbf{I}_{pt} \geq -(1-\frac{\theta^2}{2})pt+z)\right]$$

$$\lesssim e^{-(1-\frac{\theta^2}{2})(\frac{2}{\theta^2}+\delta)pt} e^{\delta(1-\frac{\theta^2}{2})pt-\delta z} = e^{-I(a,x)t-\delta z}.$$

BBM conditioned on large level set

Define $\theta := \frac{2(1-a)}{x}$, $p := \frac{(1-a)\left[x^2-2(1-a)\right]}{x^2-2(1-a)^2} > 0$ and $b := \frac{2}{\theta} = \frac{x}{1-a} > \sqrt{2}$.

Chen-M.'24+: Conditioned on large level sets

(i) (Overlap) Select two particles u_t^1, u_t^2 independently and uniformly from the $[xt, \infty)$-level set, then

$$\left(\frac{\mathcal{R}(u_t^1, u_t^2) - pt}{\sqrt{pt}} \mid L_t[xt, \infty) \geq \frac{e^{at}}{\sqrt{t}} \right) \Rightarrow \frac{\theta}{1-\theta^2/2} G$$

Under the unconditioned probability, $\mathcal{R}(u_t^1, u_t^2) = O_{\mathbb{P}}(1)$.

(ii) Let $v := bp + \sqrt{2}(1-p) > \sqrt{2}$. Then as $t \to \infty$,

$$\left(\frac{\max_{u \in \mathcal{N}_t} X_u(t) - vt}{\sqrt{t}} \mid L_t[xt, \infty) \geq \frac{e^{at}}{\sqrt{t}} \right) \Rightarrow \frac{\sqrt{2}-\theta}{\sqrt{2}+\theta} G$$

Under the unconditioned probability, $M_t = \sqrt{2}t + O(\log t)$.

Thanks for your attention!