Extrema of two-type reducible branching Brownian motions

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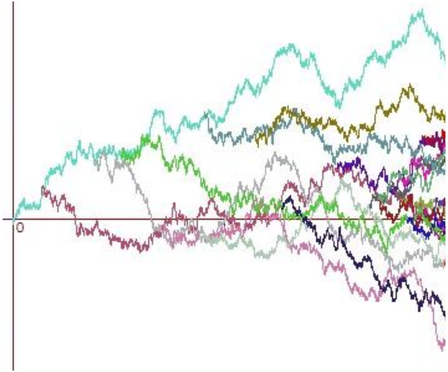
Based on joint works with Yanxia Ren (Peking University)

Branching Brownian motion (BBM)

- Initially a particle move as a Brownian motion with diffusion coefficient σ^2.
- At rate β it splits into two particles.
- These particles behave *independently* of each other, continue move and split, subject to the same rule.

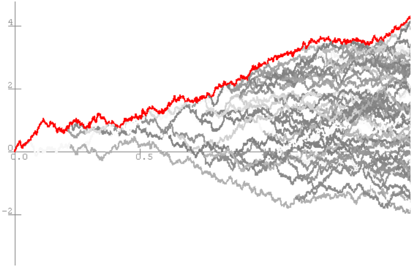


Trajectories of particles in a BBM.

Maximum of BBM

Let $\mathsf{M}_t := \max_{i \le n(t)} \mathsf{X}_i(t)$ be the maximum among all the particles alive at time t.

- **Biggins'76:** $\lim_{t\to\infty} \frac{\mathsf{M}_t}{t} = \sqrt{2\beta\sigma^2}$ a.s.



A trajectory of M_t

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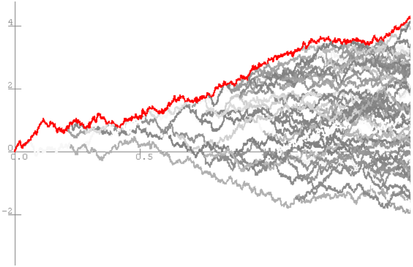

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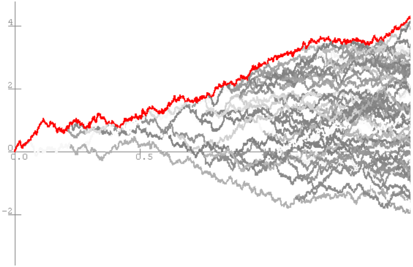
$m_t := \sqrt{2\beta\sigma^2}t - \frac{3}{2\sqrt{2\beta/\sigma^2}} \log t.$
- **Lalley-Sellke'87**: The limiting distribution is a *randomly shifted Gumbel distribution*: There exist constant C and random variable Z_∞ such that

$$\lim_{t\to\infty} \mathsf{P}(\mathsf{M}_t - m_t \le x) = \mathsf{E}[\exp\{-C\mathsf{Z}_\infty e^{-\sqrt{2\beta/\sigma^2}x}\}].$$



A trajectory of M_t

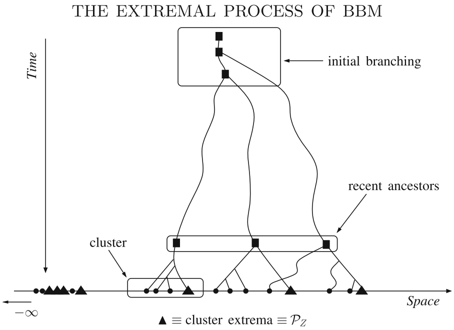
Full extremal value stasistics

Here we describe the result for standard case: $\beta = \sigma^2 = 1$.

- **Aïdékon-Berestycki-Brunet-Shi'13** and **Arguin-Bovier-Kistler'13**: The *extremal process* $\sum_{i\le n(t)} \delta_{\mathsf{X}_i(t)-m(t)}$ converges in law to a certein *decorated Poisson point process* (DPPP):

$$\sum_{i\le n(t)} \delta_{\mathsf{X}_i(t)-m(t)} \Rightarrow$$

$$\mathrm{DPPP}(C\mathsf{Z}_\infty e^{-\sqrt{2}x}dx, \mathfrak{D}^{\sqrt{2}}).$$



Construction of the DPPP

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$$\lim_{N\to\infty} P\left(\max_{v \in V_N} X_v^N \le m_N + x\right) = E\left[e^{-CZe^{-\frac{2}{\sqrt{g}}x}}\right].$$

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- ϵ-Cover times of 2D sphere by Brownian motion (**Dembo-Peres-Rosen-Zeitouni'04**, **Belius-Kistler'17**, **Belius-Rosen-Zeitouni'19**)

$$\sqrt{\mathcal{C}_\epsilon} = 2\sqrt{2}\left(\log\epsilon^{-1} - \frac{1}{4}\log\log\epsilon^{-1}\right) + O_P(1)$$

- Characteristic polynomials of random matrices, High-values of the Riemann-zeta function, $\cdots\cdots$

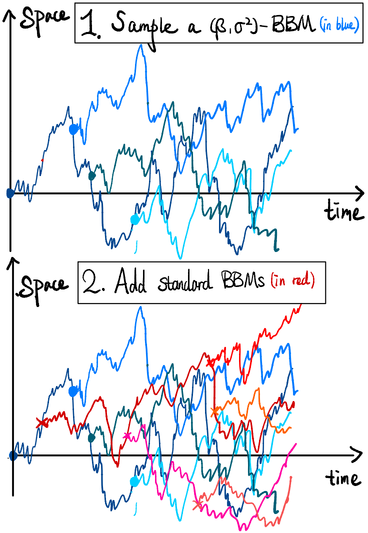
Variants of BBM are also received many attention.

- Variable speed BBM/Generalized random energy model (**Fang-Zeitouni'12**, **Bovier-Hartung'14**, **Bovier-Hartung'15**, **Mallein'15**, **Maillard-Zeitouni'16**, **Bovier-Hartung'20**)
- Multitype BBM. (Irreducible case: **Biggins'76**, **Ren-Yang'14**, **Hou-Ren-Song'23+**)
- d-dimensional BBM (**Mallein'15**, **Stasiński-Berestycki-Mallein'22**,**Kim-Lubetzky-Zeitouni'23**, **Berestycki-Kim-Lubetzky-Mallein-Zeitouni'21+**.)
- Hyperbolic BBM (**Lalley-Sellke'97**), Branching random walks on hyperbolic groups (**Sidoravicius-Wang-Xiang'22**, **Dussaule-Wang-Yang'22**)
- ......

Our model: Two-type BBM

In a two-type <u>reducible</u> branching Brownian motion:

- Type 1 particles move as Brownian motion with diffusion coefficient σ^2. They split at rate β into two children of type 1; and give brith to type 2 particles at rate α.



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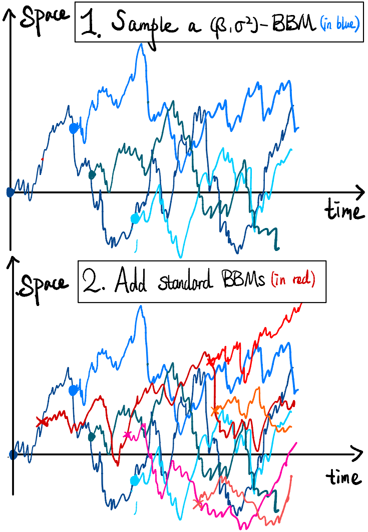
- Type 2 particles move as standard Brownian motion and branch at rate 1 into two type 2 children, but can not produce children of type 1.



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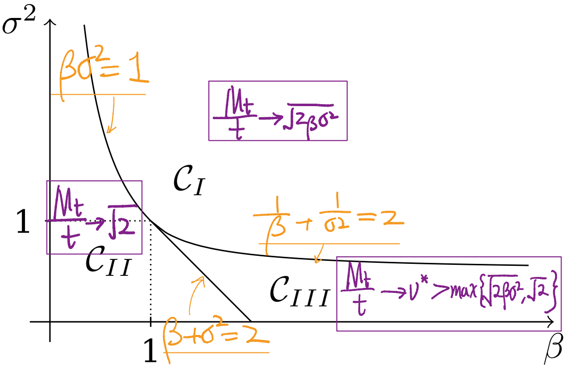
- **Asymptotic behavior of extremal particles.** One should excepe that the extremal process converges in law to certain decorated Poisson point process.
$$\sum_{i=1}^{n(t)} \delta_{X_i(t) - C_1 t + C_2 \log t} \Rightarrow \mathbf{DPPP}$$

Leading order of the maximum

Biggins'12 obtained the spreading speed $\lim_{t\to\infty} \frac{M_t}{t}$ (in a more general setting.)

- If $(\beta, \sigma^2) \in \mathscr{C}_I$ (resp. $\mathscr{C}_{II}$), type 1 (resp. type 2) particles are dominating: $M_t/t \to \sqrt{2\beta\sigma^2}$ (resp. $\sqrt{2}$) $=$ speed of BBM with single type 1 (resp. type 2).



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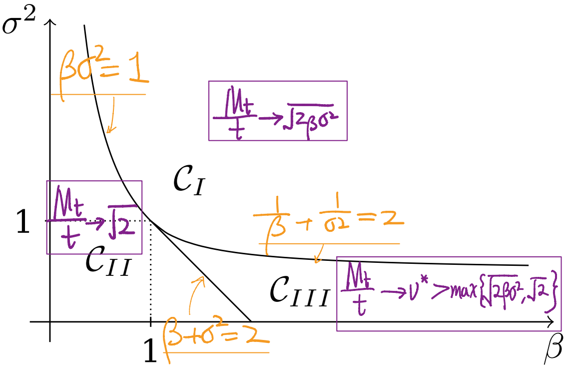
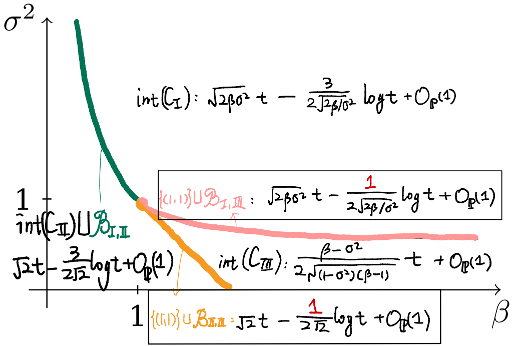
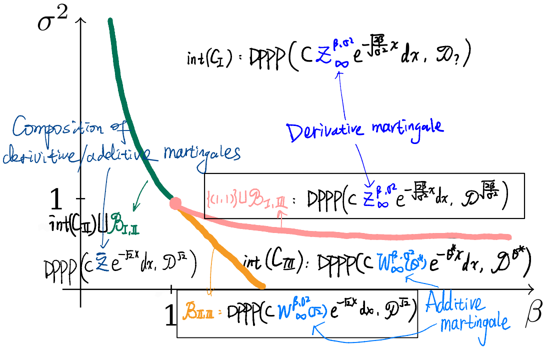
- If$(\beta, \sigma^2) \in \mathscr{C}_{III}$, $M_t/t \to v^* = \frac{\beta-\sigma^2}{\sqrt{2(1-\sigma^2)(\beta-1)}} > \max\{\sqrt{2\beta\sigma^2}, \sqrt{2}\}$. This was called anomalous spreading, as the speed of the two-type process is strictly larger than the speed of both single type particle systems.



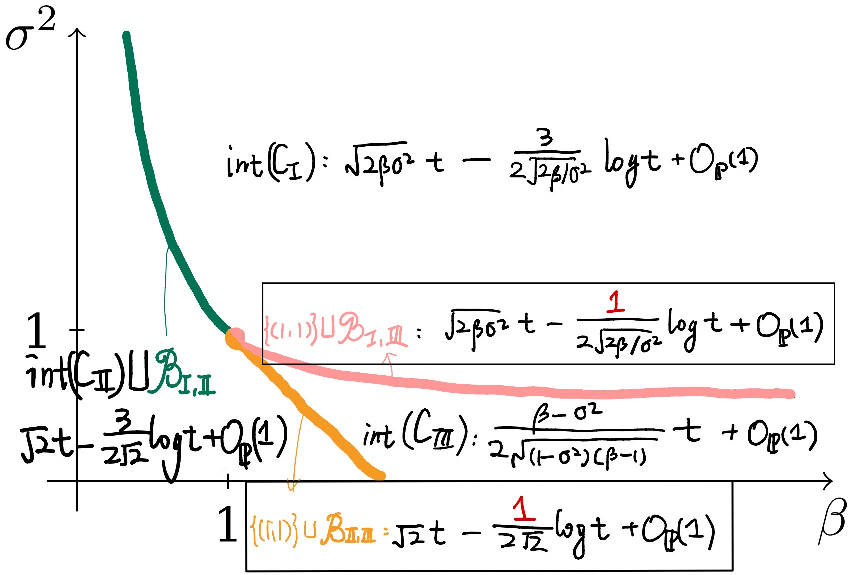
Subleading order of the maximum

Belloum-Mallein'21 investigated the subleading order of the maximum M_t and the limiting extremal processes, when the parameter (β, σ^2) are *interior points* of regions $\mathscr{C}_I, \mathscr{C}_{II}, \mathscr{C}_{III}$.
Belloum'22+ considered a special critical case $\beta = \sigma^2 = 1$.
M.-Ren'23+ investigated the case that (β, σ^2) lies on the *boundries* between $\mathscr{C}_I, \mathscr{C}_{II}, \mathscr{C}_{III}$.

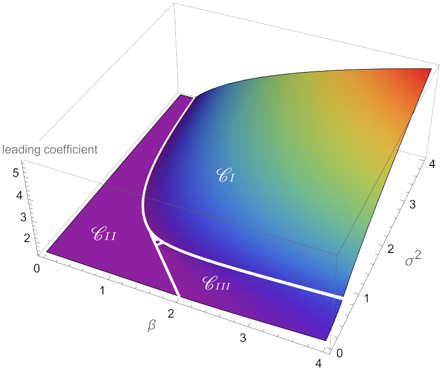


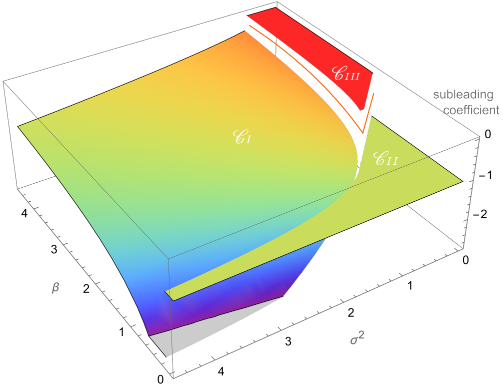


Subleading order of the maximum



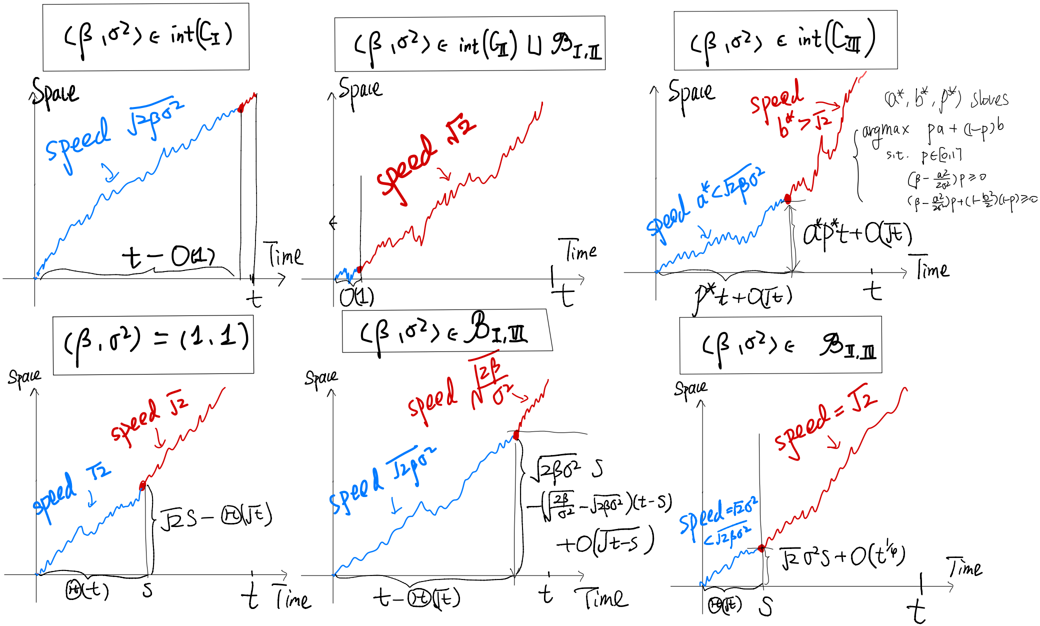
Double jump in the maximum





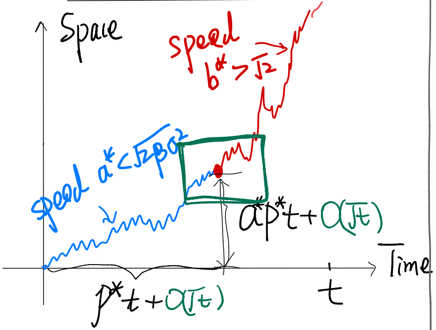
Note that a **double jump** occurs when parameters (β, σ^2) cross the boundary of the anomalous spreading region $\mathscr{C}_{III}$, and only a single jump occurs when (β, σ^2) cross $\mathscr{B}_{I,II}$.

Localization of extremal particles



Localization of extremal particles

$$\boxed{(\beta, \sigma^2) \in \mathrm{int}(C_{III})}$$



length of time window

length of space window

$$\sqrt{t} \times \left(\sqrt{t} \times \frac{1}{\sqrt{t}}\right) \times \frac{O(1)}{\sqrt{t}} = O(1)$$

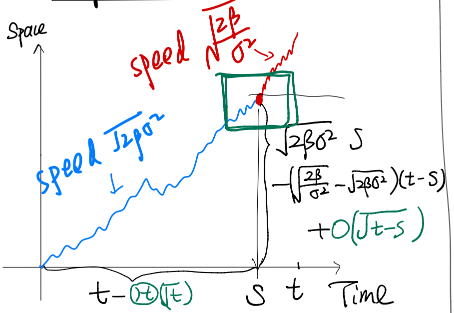
LLT LLT

$\Downarrow$

$$\boxed{\text{Coefficient} = 0}$$

Localization of extremal particles

$$(\beta, \sigma^2) \in \mathcal{B}_{I,II}$$



Length of time window

Ballot theorem

length of space window

$$\sqrt{t} \times \frac{t-s}{s} \times \frac{\sqrt{t-s}}{\sqrt{t}} \times \frac{O(1)}{\sqrt{t-s}}$$

LLT LLT

$t-s=O(\sqrt{t})$

$$= \frac{1}{t^{\frac{1}{2}}}$$

$\Rightarrow$ Coefficient $= \frac{1}{2}$

Intermediate phases: From 0 to 3

- Idea: Let $f_t(x) = x^t$, $x \geq 0$ is the parameter and $t > 0$ is the time. Then

$$\lim_{t\to\infty} f_t(x) = \begin{cases} \infty & x > 1 \\ 1 & x = 1 \\ 0 & 0 \leq x < 1 \end{cases}$$

To get a continus phase transation we set parameter x depends on time t and approach critical point 1 properly: set $x^{\pm}_{t,h} = 1 \pm \frac{h}{t}$ then

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- Inspired by **Schmidt-Kistler'15**, **Bovier-Hartung'20** we assume that parameters (β, σ^2) depends on the time horizon t and are close to the boundaries $\mathscr{B}_{I,III}$: We set

$$\frac{1}{\beta_t} + \frac{1}{\sigma_t^2} = 2 \pm \frac{1}{t^h} \quad (\beta_t, \sigma_t^2) \to (\beta, \sigma^2) \in \mathscr{B}_{I,III} \tag{H}$$

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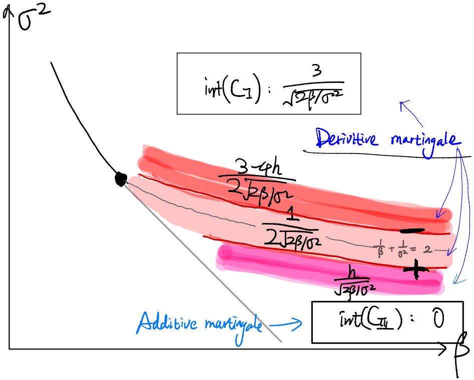
Theorem[M.-Ren, coming soon]: Define

$$m_{h,-}^{1,3}(t) = \sqrt{2\beta_t \sigma_t^2}\, t - \frac{3 - 4\min\{h, 1/2\}}{2\sqrt{2\beta/\sigma^2}} \log t$$

$$m_{h,+}^{1,3}(t) = v_t^*\, t - \frac{\min\{h, 1/2\}}{\sqrt{2\beta/\sigma^2}} \log t$$

$\{M_t - m_{h,\pm}^{1,3}(t), \mathbb{P}^{\beta_t, \sigma_t^2}\}$ converges in law. The limiting distribution is the same (up to a constant shift) as the limiting distribution of centered M_t under $\mathbb{P}^{\beta, \sigma^2}$. (Similar results hold for the extremal processes.)



Intermediate phases: From 0 to 3

Similar results are ontained for the case $\beta_t + \sigma_t^2 = 2 \pm \frac{1}{t^h}$, $(\beta_t, \sigma_t^2) \to (\beta, \sigma^2) \in \mathscr{B}_{II,III}$ or the case approaching $(1,1)$: $\frac{1}{\beta_t} + \frac{1}{\sigma_t^2} = \beta_t + \sigma_t^2 = 2 + \frac{1}{t^h}$; $\beta_t = \sigma_t^2 = 1 - \frac{1}{t^h}$; $\frac{1}{\beta_t} = \frac{1}{\sigma_t^2} = 1 - \frac{1}{t^h}$.