Shotgun threshold for sparse Erdős-Rényi graphs

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Based on the joint work with
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Identifying graphs

- **Reconstruction Conjecture** (Kelly, Ulam, Harary' 57): Any two graphs on 3 or more vertices that have the same multi-set of vertex-deleted subgraphs are isomorphic.

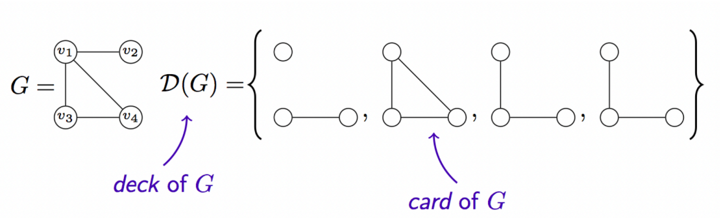

Figure 1: From Topology and Combinatorics Blog by Max F. Pitz

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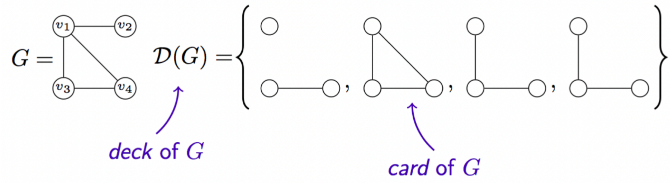

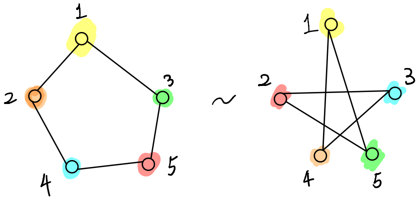

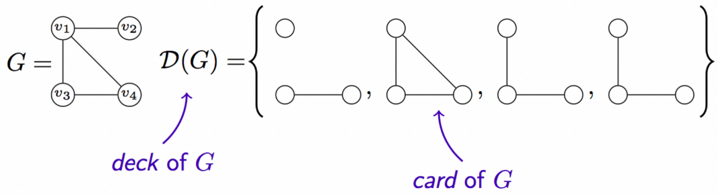

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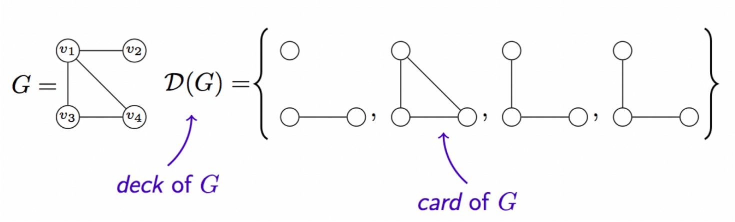

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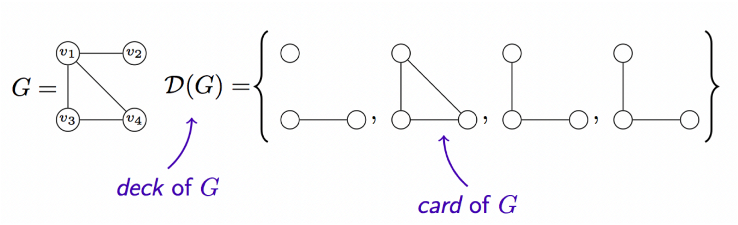
Bollobs'90: almost all graphs can be reconstructed from any 3 vertex-deleted subgraphs.



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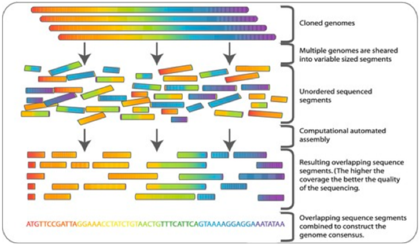
Bollobs'90: almost all graphs can be reconstructed from any 3 vertex-deleted subgraphs.

If the graph is random, but we are only given the very local information of each vertex, can we still identify the graph?

Motivating examples

- DNA shotgun sequencing: Reconstruct a DNA sequence from "shotgunned stretches of the sequence.



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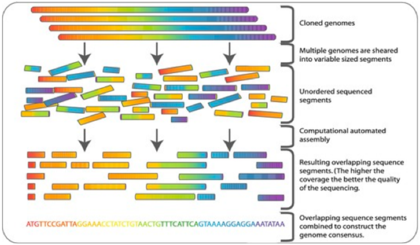

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- **Paninski et al’15**: Reconstruct a big neural network from very local subnetworks that are observed in experiments.

Mathematical framework by Mossel-Ross'19

- Model: $\mathcal{G}$ is a (fixed or random) graph, possibly with random labeling of the vertices.
- Observation: For each vertex v, we are given its local r-neighborhood $\mathsf{N}_r(v)$: the subgraph induced by the vertices (forgetting their names) at distance no greater than r from v.

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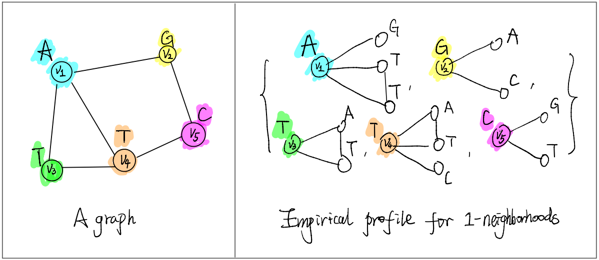

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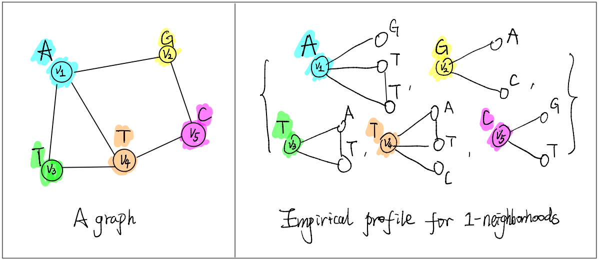

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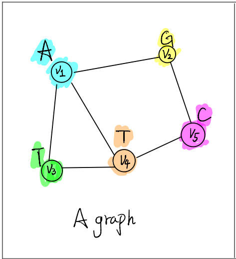
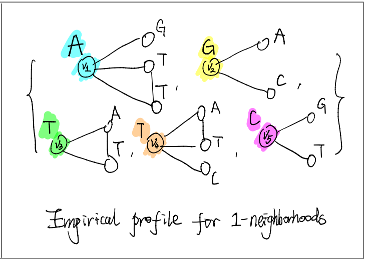
- There is a shotgun (assembly) threshold r_* for the radius r since the monotonicity.

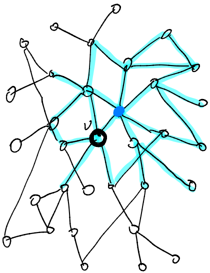




Mossel-Ross'19:

- Identifiability: Uniqueness of overlaps

$$r_* \leq \min\{k : N_k(u) \neq N_k(v)\ \forall\, u, v\} + 1$$



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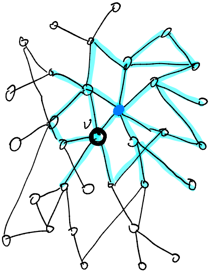

- Non-identifiability: Blocking configurations.

Labeled lattice models

Graph: d-dimensional box of side length n, denoted as Λ_n.

Labels: i.i.d. uniform vertex labels from $\{1, \cdots, q\}$.

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Ding-Liu'22+:

$$(1-\epsilon)\frac{2}{\log q}\log n \le r_* \le (1+\epsilon)\frac{2}{\log q}\log n \quad \text{when } d = 1;$$

$$(1-\epsilon)\frac{d}{\log q}\log n \le (r_*)^d \le (1+\epsilon)\frac{d}{\log q}\log n \quad \text{when } d \ge 2.$$

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Indeed, if $r \geq (0.5+\varepsilon)\log_{d-1} n$ then for all $u \neq v$, $(d_1(v), \ldots, d_r(v)) \neq (d_1(u), \ldots, d_r(u))$ where $d_i(v)$ are the number of nodes at distance i from v.

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- (Almost) all $0.5\log_{d-1}(n)$ neighborhoods are trees.
- However, each $0.5(1+\epsilon)\log_{d-1}(n)$ neighborhoods is encoded by a unique cycle structure.

Erdős-Rényi graphs (dense regime)

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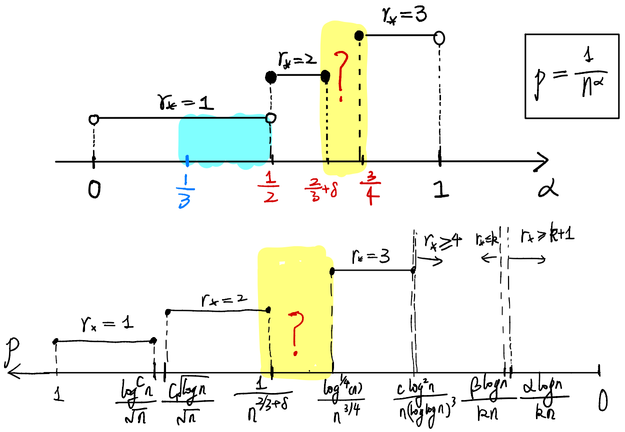
Mossel-Ross'19, Gaudio-Mossel'20, Huang-Tikhomirov'21+, Johnston-Kronenberg-Roberts-Scott'22+:



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- The lower bound also HOLDs for $\lambda = 1$.
- $C_\lambda = \frac{1}{\log(\lambda^{-1})}$ when $\lambda < 1$ and $C_\lambda = \frac{1}{\log(\lambda)} + \frac{2}{\log(1/\lambda_*)}$ when $\lambda > 1$, where $\lambda_* < 1$ satisfies $\lambda e^{-\lambda} = \lambda_* e^{-\lambda_*}$.

Shotgun threshold for $\mathcal{G}_{n,\frac{\lambda}{n}}$

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Theorem (Ding-Jiang-**M**. 22+)

Fix $\lambda \in (0,\infty)$. Let $\gamma_\lambda = \mathbb{P}(\mathbf{T} \sim \mathbf{T}')$, where $\mathbf{T}, \mathbf{T}'$ are two independent PGW(λ) trees. Let $\mathcal{G} \sim \mathcal{G}\left(n, \frac{\lambda}{n}\right)$. Then for any $\epsilon > 0$, w.h.p.,

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- Indeed there is a power series A with non-negative coefficients such that $\lambda^2\gamma_\lambda = A(\lambda e^{-\lambda})$.
- We also give an algorithm with polynomial running time for reconstructing the original graph.

Non-identifiability: blocking configuration

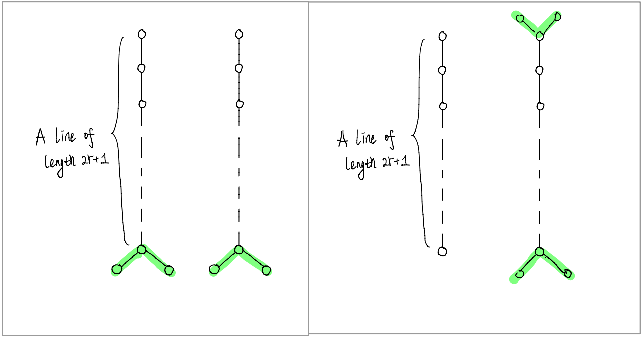

Figure 3: Blocking subgraph by Mossel and Ross

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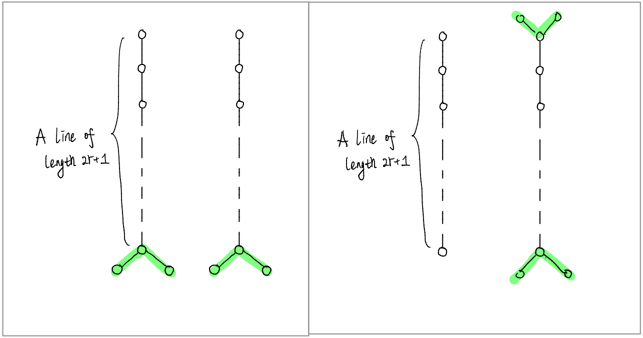

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Letting the expactation of the number of such blocking subgraphs $n^2 \times (\lambda e^{-\lambda})^{2r} \times (\lambda e^{-\lambda})^{2r} \geq 1$, we have $r \leq \frac{1}{2(\lambda - \log \lambda)} \log n$.

Non-identifiability: blocking configuration

Key: The middle part (2r levels) are isomorphic; in addition removing red vertices results in small bushes

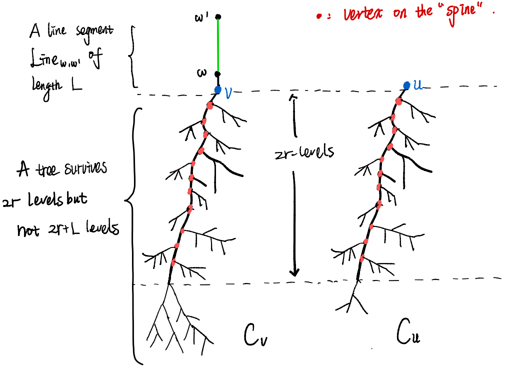
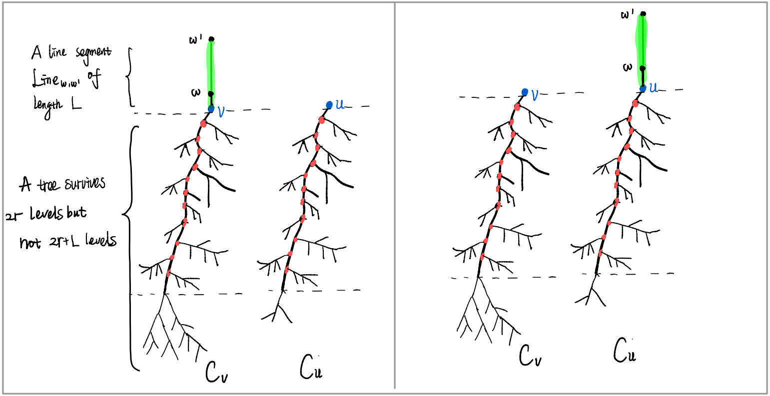


Figure 4: Improved blocking subgraph

Non-identifiability: blocking configuration



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Lemma (Ding-Jiang-M. 22+)

$\mathbb{P}(\mathbf{T} \sim_{2r} \mathbf{T}') \asymp \alpha_\lambda^{2r}$ *where* $\alpha_\lambda := \lambda^2 \gamma_\lambda < 1$.

Letting $n^2 \times \alpha_\lambda^{2r} \geq 1$, we need $r \leq \frac{1}{\log(\alpha_\lambda^{-1})} \log n$.

Brief discussion on identifiability

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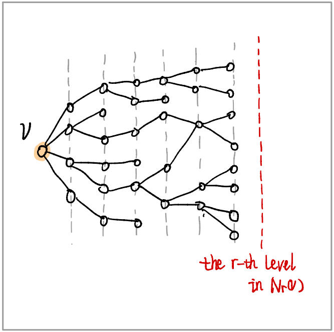

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We know that $N_r(v)$ is a component!

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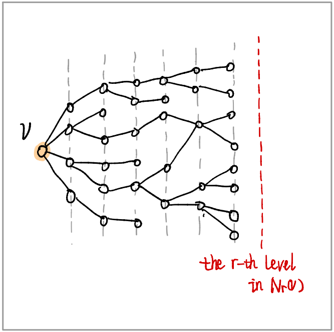
- **uczak'98, Riordan-Wormald'10**: each connected component has diameter less than $C_\lambda \log n$ when $\lambda \neq 1$.
- **Nachmias-Peres'08**: the diameter is of order $n^{1/3}$ for $\lambda = 1$.



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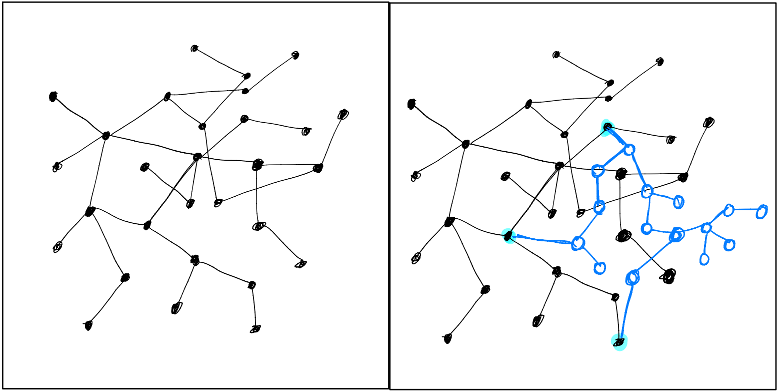
Our Intuitions

Our key intuition is to recover bad components from good vertices.



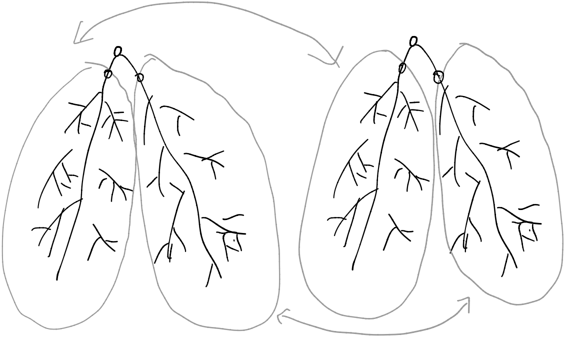
● : good veritices ○ : bad vertices

Brief discussion on identifiability

Assume that $r = \frac{1+\epsilon}{\log(\alpha_\lambda^{-1})} \log n$.

Key observation 1: Vertices which have **two disjoint r-arms** in their r-neighborhood are good:

- "Essentially", their $(r-1)$-neighborhood is unique, since $n^2 \times \alpha_\lambda^{2r} \ll 1$.



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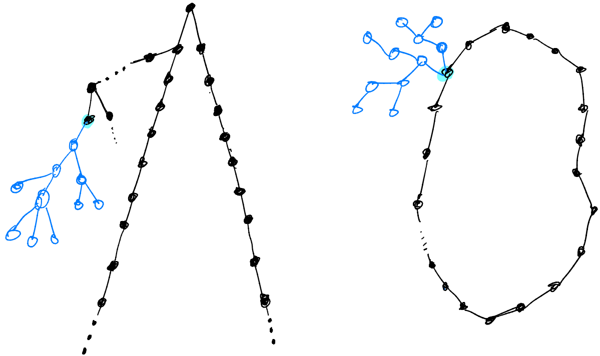
Key observation 1: Vertices which have **two disjoint r-arms** in their r-neighborhood are good.

- Vertices without two r-arms can be identified from the r-neighborhood of some good vertex (or it is in a small component).



Brief discussion on identifiability

Caveat: we have **ignored cycles** in the graph in our analysis above, and this incurs serious challenge.

Key observation 2: Vertex which is contained by some cycle but doesn't have unique $(r-1)$ neighborhoods are very rare.

Thanks for your attention!